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THE NOVEMBER MEETING OF THE IOWA SECTION.

The eleventh regular meeting of the Iowa Section was held at the West Des Moines High School, Des Moines, Iowa, on November 3, 1922, the Section, as is usual at the fall meeting, uniting with the Iowa Association of Mathematics Teachers. The meeting consisted of one session with Professor R. B. McClenon, president of the Iowa Association, in the chair.

There were about one hundred and fifty in attendance, including the following twenty-one members of the Association:

O. W. Albert, R. P. Baker, I. S. Condit, Marian E. Daniells, R. M. Deming, F. C. Earhart, C. W. Emmons, Iva Ernsberger, Fay Farnum, C. Gouwens, G. E. King, R. B. McClenon, J. V. McKelvey, Martha McD. McKelvey, I. F. Neff, J. F. Reilly, H. L. Rietz, Maria M. Roberts, E. R. Smith, G. W. Snedecor, C. W. Wester.

The next meeting will be held at Cornell College, Mount Vernon, Iowa, April 27–28, 1923, in conjunction with the annual meeting of the Iowa Academy of Science.

The following two papers were read:

(1) "The appreciation of four dimensions" by Professor R. P. BAKER, University of Iowa;

(2) "How can we improve the teaching of mathematics in Iowa" by Miss WHEELER, teacher of mathematics, Grinnell High School; and Professor E. R. SMITH, Iowa State College.

A general discussion on the improvement of mathematics teaching followed the presentation of these papers, abstracts of which follow below, the numbers corresponding to the numbers in the list of titles:

1. In this paper Professor Baker discussed Poincaré's suggestion that by making independent our two methods of judging depth—shading and binocular convergence—an appreciation of four dimensions might be reached. A set of stereoscopic pictures was shown illustrating retinal rivalry where the compromise between the judgments of two eyes is an alternation in time; and another set where, by putting the usual right and left eye pictures in opposite places, the shading and binocular convergence work against one another. In this case the compromise is by majority rule of associations. The standard method of models in three dimensions of projections of four-dimensional solids was discussed and illustrated by models of the tesseract and the square on the triangle.

2. To improve the teaching of mathematics Miss Wheeler placed emphasis on the following three points: (*a*) the teacher should feel that the subject taught

is worth while, (*b*) the mathematics department should be supplied with equipment as liberally as is any other department, and (*c*) the individual student should be kept in mind and the course adapted as far as possible to his needs.

Professor Smith asserted that at the present time the Colleges and Universities of the State are supplying annually only about fifty per cent. of those required to meet the demand in the high schools and in the industries. He recognized four groups of students studying mathematics in the high schools: (*a*) those taking tradesmen's courses, (*b*) those taking such courses as will prepare them to read current literature, (*c*) those who will become technicians, and (*d*) those who will be mathematics specialists. In order to improve the instruction given in the high school, especially for the latter two groups, he would like to see a larger number of ideal teachers whose preparation consisted of not less than four hours per week for four years in College, and one year of professional work in teaching.

J. F. REILLY, *Secretary-Treasurer.*

THE NOVEMBER MEETING OF THE MISSOURI SECTION.[1]

The sixth regular annual meeting of the Missouri Section was held at the Junior College of Kansas City on Saturday morning, November 18, 1922, in connection with the annual meeting of the Missouri State Teachers' Association. Professor W. A. Luby, vice-chairman of the section, presided. Mr. Alfred Davis acted as secretary in the absence of Professor P. R. Rider.

There were twelve in attendance, including the following six members of the Association:

L. V. Cutting, Alfred Davis, S. Lefschetz, W. A. Luby, A. D. Pierson, R. A. Wells.

The present officers were elected for another year. They are Professor E. R. HEDRICK, University of Missouri, chairman; Professor W. A. LUBY, Junior College of Kansas City, vice-chairman; Professor P. R. RIDER, Washington University, secretary-treasurer.

The following papers were presented:

(1) "A study of the data determining the sun-spot maximum of 1829" by Professor W. A. LUBY, Junior College of Kansas City;

(2) "Problems concerning the teaching of secondary mathematics" by Mr. ALFRED DAVIS, Soldan High School, St. Louis;

(3) "Mathematics in Europe" by Professor S. LEFSCHETZ, University of Kansas (by invitation).

Abstracts of papers follow below, the numbers corresponding to the numbers in the lists of titles:

1. The observations on which the date of the sun-spot maximum of 1829 is based are those of Schwabe, Stark and Arago. Professor Luby's paper presented

[1] The secretary of the section is indebted to Mr. Alfred Davis for the material for this report

an abstract of the observations and pointed out the smaller weight given to the more reliable data of Schwabe.

2. Mr. Davis's paper stated that recent activities among some of the prominent educators of the country, with reference to mathematics in our high schools, have emphasized some problems and have created other problems that need solution. Some of these have been partially solved by such agencies as the National Committee on Mathematical Requirements; by special schools, such as the Horace Mann School, New York; and by individual teachers. Much however remains to be done. And their complete solution, let us hope that this may not be long delayed, will help to restore mathematics to its proper place in the class room, as well as outside. Mathematics is developing with great rapidity along many lines and it is becoming increasingly important as an aid to progress in many branches of science. In view of this and of the educational value of the subject we need to define more clearly, and without any trace of apology, the aims and purposes of the subject. Much time is wasted and vantage lost because pupils are not classified according to ability to learn. The dull are overwhelmed and the bright are retarded. Not only do the junior high school courses need to be more clearly outlined, but, and no less important, the senior courses need re-organizing. The traditional courses have served their day. Pupils are not now merely prepared for college; the majority of those who graduate from high school do not go to college. All plane and solid geometry beyond the junior high school might be condensed into one year. Similarly all the algebra beyond that given in junior high school might be taught in one year. This leaves the senior year for some advanced courses. The latter may consist of trigonometry, of history of mathematics, or advanced work which would include the elements of analytics and the calculus. By such means the pupil might be given a working knowledge of mathematics and so have the field for further study and application of the subject opened to him. The standing of the subject among people generally could be greatly improved by giving the brighter of our high school pupils this advanced work. Mathematics would thus seem to have a real purpose in the work-a-day world. The climax of our problems is the securing of the inspired and the inspiring teacher. Let us remember that every good cause needs championing by its friends. If these problems are solved it will be because we solve them. Indifference must vanish.

3. Professor Lefschetz recited some of the impressions gathered during his recent trip and spoke of the outlook, which is anything but brilliant, for science in general, and for mathematics in particular. (See *1922*, 144.)

P. R. RIDER, *Secretary-Treasurer.*

THE DECEMBER MEETING OF THE MARYLAND–VIRGINIA–DISTRICT OF COLUMBIA SECTION.

The twelfth regular meeting of the Maryland-Virginia-District of Columbia Section was held at the Bureau of Standards, Washington, D. C., on Saturday, December 9, 1922. The Chairman of the Section, Professor Frank Morley, presided at both morning and afternoon sessions. At the close of the morning session, those in attendance at the meeting had dinner in the dining-hall of the Bureau of Standards as guests of the Washington members of the Section. The hour following dinner was spent in an inspection of some of the work being carried on at the Bureau of Standards.

The next meeting of the section will be held on May 12, 1923, at Baltimore, Maryland.

There were fifty-eight in attendance, including the following thirty-seven members of the Association:

O. S. Adams, J. J. Arnaud, R. N. Ashmun, H. G. Avers, Sarah Beall, G. A. Bingley, W. H. Bixby, C. C. Bramble, J. A. Bullard, G. R. Clements, A. Cohen, F. W. Darling, C. H. Davis, A. Dillingham, H. English, J. B. Eppes, W. M. Hamilton, W. E. Heal, W. D. Lambert, A. E. Landry, E. A. LeLacheur, Florence P. Lewis, Nannie J. McKnight, F. Morley, C. A. Mourhess, F. D. Murnaghan, J. R. Musselman, C. A. Nelson, E. C. Phillips, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, H. M. Robert, Jr., H. A. Robinson, R. E. Root, C. A. Shook, E. W. Woolard.

The following papers were read:

(1) "A problem in the theory of numbers" by Dr. J. R. Musselman;

(2) (*a*) "A remarkable formula for prime numbers," (*b*) "A method of distinguishing between prime and composite numbers of large size" by Dr. Paul R. Heyl, Bureau of Standards (Introduced by Mr. W. D. Lambert);

(3) "Remarks on the proposed plan of reorganization of secondary school mathematics" by Professor A. E. Landry;

(4) "Dynamic symmetry" by Professor H. M. Robert, Jr.;

(5) "Remarks on a problem in geometry" by Professor Frank Morley;

(6) "A property of a system of partial differential equations" by Dr. C. A. Nelson.

Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Dr. Musselman discussed the problem of finding even "perfect" numbers, which reduces to finding values of $2^n - 1$ which are primes. He also mentioned what is known concerning the expression of a number as a sum of cubes.

2. Dr. Heyl uses the expression $(2^{n-1} - 1)/n$, which is integral for all odd *prime* values of n, and non-integral for all even values of n. Computation shows that it is also non-integral for all composite odd values of n up to 1,000 with the exception of 341 and 645. A practical method of handling the very large numbers to which this formula gives rise may be based on the device of expressing large

powers of 2 as powers of powers; and the making use of the principle that if the product of several numbers be divisible by n with a certain remainder, this same remainder will be obtained if for any of these numbers there be substituted its remainder when divided by n. The remainder resulting from this formula will in about 75 per cent. of the cases examined (1,000 in number) contain a factor of the original number, which may then be found by the usual process for highest common factor. Time tests have shown that a number of the order 10^8 may thus be reduced in one hour as against 20 hours required for the trial of prime divisors.

3. Professor Landry's paper was an exposition and criticism of the plan of reorganization of secondary school mathematics drawn up by the National Committee on Mathematical Requirements, and printed in part by the United States Bureau of Education in 1921 as Bulletin No. 32. Attention was devoted chiefly to Chapters III and IV, on the junior and senior high school periods respectively. The proposed junior high school course was on the whole approved except as to (1) the extent to which unification is possible or desirable, and (2) a profitable lessening in the amount of *time* allotted to algebraic drill. The suggestion that calculus be introduced as an elective in the senior high school course was examined and approved, but with the accompanying expression of serious doubt of the likelihood of its early adoption.

4. Professor Robert discussed the theory of Mr. Jay Hambidge regarding the proportions of Greek vases as explained in his book *Dynamic Symmetry*, and in *Geometry of Greek Vases* by Dr. L. D. Caskey, Curator of Classical Antiquities of the Museum of Fine Arts, Boston. Mr. Hambidge finds that the Greeks used geometry rather than arithmetic in design. Rectangles having the ratio of longer to shorter side equal to $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, are called root-two, root-three, and root-five rectangles, respectively. These rectangles are combined with the square to form other rectangles. A rectangle derived from the root-five rectangle, having the ratio $\frac{1}{2}(\sqrt{5}+1)$ is called the whirling square rectangle and is much used. Mr. Hambidge finds that if the over-all proportions of a vase, that is, the ratio existing between its height and greatest width, is expressible in terms of one of these three root-rectangles, then the heights and widths of all its parts can be expressed in terms of that rectangle and of no other; *e.g.*, root-two or root-three never appear in connection with root-five. In Dr. Caskey's book dimensions to the nearest half-millimeter of most of the vases in the Museum are given. The outline of each vase is enclosed in a rectangle belonging to one of the systems of dynamic symmetry, the margin of error allowed averaging less than one millimeter. The interrelation of details is shown by subdivision of the containing rectangles and by intersections of diagonals.

Professor Robert gave illustrations of the analysis of vases and pointed out misunderstanding on the part of critics and failure to recognize that all constructions must be geometrical. In particular, he discussed the suggestion in a published criticism (this MONTHLY, *1922*, 164) that .9393 might be mistaken for .927. He showed the drawing for the kylix in question and the simplicity of the

geometrical construction of the diameter of the foot as $2 - 3\sqrt{2}/4 = .9393$ in relation to the over-all rectangle.

5. Professor Morley made some remarks on recent work on the three-bar curve, especially on the memoir on the analytical treatment of the three-bar curve by F. V. Morley in *Proceedings of the London Mathematical Society*, series 2, volume 21, pp. 140–160.

6. Dr. Nelson pointed out the relation between the solutions of a completely integrable system of linear partial differential equations of the second order and of the adjoints, in the sense of Riemann, of these equations.

G. R. CLEMENTS, *Secretary-Treasurer.*

THE ORGANIZATION OF COLLEGE COURSES IN MATHEMATICS FOR FRESHMEN.[1]

By J. W. YOUNG, Dartmouth College.

The movement toward so-called unified courses in mathematics for freshmen reached a certain rather definite stage of growth with the publication in 1914 of Professor Slichter's *Elementary Mathematical Analysis*. It is true that before this date other texts embodying a breaking down of the traditional barriers between various subjects had appeared, such as the well-known texts of Smith and Granville, Woods and Bailey, Brenke, Ziwet and Hopkins. But these were either intended for special classes of freshmen, or for sophomores, or they involved the correlation merely of two specific subjects, such as algebra and trigonometry or algebra and analytic geometry.

Professor Slichter's text was, as far as I know, the first that attempted to give a unified course for the entire freshman year based on the minimum college entrance requirements. It is rather interesting to note, in passing, that this first attempt was written specifically for students preparing to study engineering rather than for the general student. This text was followed by the texts of Young and Morgan (1917), Karpinski, Benedict and Calhoun (1918), Webber and Plant (1919), Gale and Watkeys (1920) and Griffin (1921). I note that the next speaker is to discuss the historical aspects of this movement. I do not wish to trespass on his domain; this much, however, seems to be necessary as an explanation of what follows.

It seemed to me that it might be of interest to try to get some idea as to the extent to which the new type of course has been adopted. I accordingly appealed to the publishers of the textbooks listed in the preceding paragraph with the purpose of securing a list of colleges and universities using the various texts in the various years since their publication. These publishing firms were all very cordial in their replies and furnished me with such data as they had—they pointed out, however, that their data were very unreliable for my purposes. I accord-

[1] Read at the meeting of the Association, University of Rochester, Sept. 6, 1922.

ingly sent a questionnaire to certain colleges and universities selected as follows: I included first all institutions listed in the World Almanac for 1922 as having an endowment of at least $1,000,000; I then added all the larger state Universities and Colleges, and then selected other institutions having at least 1,000 students. This gave a list of 165 institutions. The questionnaire was sent out Aug. 4, when many of those addressed were away from their institutions. It is, therefore, it seems to me, a fair showing that replies were received from 98 institutions;[1] of these apparently 59 have at one time or another within the last ten years given some form of unified courses to freshmen. If we may assume that the 98 institutions which replied form a typical selection from the institutions of the country, it is clear that the new type of course has been at least tried in a very substantial proportion of our institutions. The lists furnished me by the publishers indicate that seven of the eight textbooks listed (no list was secured from one of them) have been used in about 150 different institutions in this country and Canada. It would seem that the time has come when a critical estimate of the results secured by unified courses could be made with profit. The combined experience of all these institutions would be extremely valuable could it be secured and evaluated.

We must be content today, however, with a less ambitious program—one that is commensurate with our limited data and with the limited time at our disposal.

Of the 59 institutions which have given unified courses, 41 gave detailed information as to the years in which certain texts were used. From these data it is possible to derive an answer to the question whether the movement toward a unified course for freshmen is at present on the increase or the wane. Of these 41 institutions, 14 were giving the new type of course in 1917–18, 20 were giving such a course in 1918–19, 27 in 1919–20, 23 in 1920–21, and 23 in 1921–22. These figures may indicate a slight decrease in enthusiasm during the past two years; but the data are very incomplete and too much reliance must not be placed on them. The figures do show conclusively that a strong body of institutions still believe in the new type of course—and that those who are interested actively in furthering the cause have no need to feel discouraged.

Those of us, however, who believe in the new type of course must by no means feel that the battle is won—or that our work is done. The opposition is strong, the criticisms severe and often just, the problem of constructing the ideal freshman course is far from solved—and the solution presents many very serious difficulties. It is still to my mind an open question whether these difficulties can be overcome.

The replies to my questionnaire have been very helpful in my attempt to determine and evaluate these difficulties. A brief summary of the arguments against unified courses will indicate some of these difficulties.

A number of arguments relate to administrative problems. Transfer of credits from one institution to another is difficult; the unified course is not well

[1] The figures here given differ slightly from those presented at Rochester, since I have included replies received since that time.

adapted to institutions operating on the term or quarter plan; at many institutions students can not be counted on to take mathematics for a whole year; the course is not well adapted to the varied preparation with which freshmen enter college, etc. With such difficulties I do not propose to deal today; not because they are not real or important, but because their discussion would take us too far afield. Perhaps some of the other speakers will touch upon them.

A number of other arguments relate to the needs of special classes of students, such as prospective engineers, scientists, agriculturists, etc. These also I shall pass over without comment, as I think it desirable to limit my own discussion to a fairly definite problem.

Before proceeding further with the enumeration of difficulties, let me attempt to define this problem with reasonable precision:

I have in mind a student who enters college with the mathematical preparation implied by a year of plane geometry and a year or a year and a half of elementary algebra. I realize that students especially in our eastern colleges often have somewhat more preparation; above all I realize that in our western institutions large numbers enter with less. But I conceive the preparation indicated to be the natural starting point for our discussion. I will suppose furthermore that this student expects to take one full year of mathematics, three hours per week, during his freshman year. I assume, furthermore, that he does not know what his life work is to be, nor in what department of college he expects to specialize. What is the best course in mathematics which we can plan for this student? That is the problem I wish to approach and to which I propose to limit my own part in this discussion. I venture to believe that the problem as thus formulated is the central problem from the solution of which other modified problems will have to result.

I return to the difficulties in the way of a solution. One of the most insistent claims made by the adverse critics of a unified course is to the effect that it does not produce clear-cut concepts in the mind of the student, that he is confused by the multiplicity of ideas and methods presented to him. This is a serious defect of the course, if it is a fact—and we must remember that I am giving you the testimony of men who have tried out the unified course for one or more years.

Another group of critics claims that the new type of course offers a poor foundation for later more advanced work in mathematics, that students taking it are weak in manipulation, that the course lacks thoroughness, that it is sketchy, superficial, provides only a smattering of various topics, that at the end of the course the student does not feel that he has mastered anything. This also is a serious charge—for our typical student *may* want to take more mathematics, he may want to go into a profession where he needs mathematical proficiency—and in any case none of us advocating the use of the new type of course can rest easy under the charge of superficiality and lack of thoroughness.

Another group of criticisms states that the textbooks now on the market present too much material for a year's course, that the program is too ambitious.

And finally we find serious argument to the effect that the goal set by the advocates of a unified course is in the nature of the case unattainable, that the

unifying principle fails to unify, that the desired correlation has not yet been and probably can not be made, that, as one correspondent puts it, the attempt to secure "the larger unity is at the expense of more elementary and fundamental unities without which the larger is almost unattainable."

Are we here facing a fundamental obstacle? Is our goal contrary to the laws of the learning process and of mental growth? Are we attempting the impossible?

Very many of my correspondents state that they are theoretically in favor of a unified course but that in practice it does not seem to work as well as the older traditional courses, that the results secured by the new type of course are not as good as those secured by the old. Many of these critics seem to have reached the conclusion that it is better policy to teach the separate subjects separately and then at the end attempt a general unification, that courses in the separate subjects are satisfactory if only the teacher does his part properly—that it is the teacher's duty to call attention to the various interrelations and interdependencies of the different subjects, and that this is the psychologically natural way to secure an appreciation of whatever underlying unity exists.

This leads us, as it seems to me, directly to the crux of the argument. The critics I have been quoting are arguing for the most part in favor of the traditional arrangement of the traditional material—trigonometry, algebra, analytic geometry—and against a new, unified arrangement of this traditional material. With respect to this traditional content of the freshman course they may be right, their arguments are strong, based as they are on experience, although personally in spite of manifest difficulties I have faith in the possibility of a so-called unified arrangement that will give better results than the separate-subject arrangement.

But the problem that faces us today concerning the needs of the typical general student I have described has passed entirely beyond this stage. At the present time *it is the traditional selection of material itself that is challenged.* The problem of unification is no longer the primary problem—it is a secondary problem forced on us by a new standard for the selection of content.

Most teachers, who have given serious thought to the needs of my general student, are I believe convinced that the traditional courses in trigonometry, college algebra, and analytic geometry can no longer be justified as best meeting his needs.

Can we formulate these needs? Recognizing that he probably will not take more than one year of mathematics in college, but keeping in mind the fact that he may do so and that hence we should provide a good foundation for possible subsequent courses, what would we like him to secure in the way of mathematical information and training during his freshman year?

I shall attempt to formulate these needs, as I see them. Before doing so, however, I want to say that these needs when enumerated are far beyond what can be accomplished in a one-year course. The topics which one would naturally select to satisfy these needs make a list entirely too long and extensive. It seems to me desirable, nevertheless, to list them, in order that we may have before us in all its difficulty the problem of selection that still remains. Furthermore, the

time at my disposal makes it necessary for me to be very brief. I can only indicate the general line of my thought and must count on my hearers to supply the details of the argument.

With these preliminary remarks, I now proceed to formulate the needs of my general student as follows:

1. *I should want this student to gain such mathematical information and training as will be of most use to him in later college courses in the physical and social sciences and in his later life as a citizen, parent, and educated man, without reference to his vocation or profession.*

To satisfy this demand as to the physical sciences our course must include the elements of trigonometry, linear and quadratic functions and equations, graphs, proportion and variation, familiarity with formulas and their use, and should certainly include if possible the fundamental ideas of the calculus and their applications. As to preparation for the social sciences (economics, etc.) we should want to include the elements of statistical methods, the elements of the mathematics of finance and investment, as well as some of the topics previously listed. As to his needs as citizen, parent, etc., we should again include a number of the topics just mentioned, notably statistics and the mathematics of investment, which are fundamental in many questions of public affairs, and would also want him to be familiar with the elements of probability, which in connection with the laws of compound interest are necessary for any adequate understanding of problems of insurance, to say nothing of games of chance.

2. *I should want him to gain a clear working knowledge of the fundamental general concepts in terms of which the quantitative thinking of the world is done.*

This aim to a large extent duplicates the preceding but it is broader in that it is not only utilitarian but also interpretative. Some of the concepts which I have in mind are number, ratio, measurement, congruence, similarity and proportionality, functional dependence in general, rates, limits, and so, again, the ideas of the calculus. It is under this aim that the value of training in functional thinking makes itself most strongly felt.

3. *I should want him to secure from his study of mathematics certain norms, standards and ideals of logical rigor, precision of thought, logical structure of a science or body of knowledge.*

This aim appears to me of very great importance—and here we doubtless have one of the strong arguments in favor of the separate-subject treatment, at any rate until such time as we can organize the new type of course in such a way as to exhibit a clear logical structure. I should like to go even further, however. I should like my general student to have some conception of the meaning of a logical demonstration, an understanding what is meant by a deductive system, an appreciation of the significance of assumptions, axioms, postulates, of what is meant by a formal deductive proof. To satisfy this demand we should apparently have to include in our course something concerning the foundations of mathematics—either of algebra or of geometry or of both. This leads me, however, to my fourth aim.

4. *I should want my typical student to gain some appreciation of the rôle that mathematics has played and continues to play in the development of civilization, in its material, scientific, intellectual, philosophical and spiritual aspects.*

All of the topics thus far listed can be made to serve this end. But it seems to me that we must not ignore the fact that mathematics has something fundamental to offer to the cultural development of the individual in those topics, of rather recent development, which have close contact with philosophy and possibly with religion. I refer to the various developments relating to the concept of infinity and the theory of classes, non-euclidean and n-dimensional geometry, etc. The proposal to include such material in an elementary course must appear very fanciful—and I do not myself see how it can be done in a one-year course. But interest in these topics is great and widespread among educated people—and it seems to me that one of the important higher functions of education should be to minister to such interests. Is it not precisely by furnishing the means of satisfaction to such latent interests that the general intellectual level can be raised? It seems to me altogether proper, therefore, that in a discussion of this sort we should at least recognize the existence of this need and appreciate its importance.

If you agree with my formulation of needs for the general student, I think you will have to agree that the traditional content of trigonometry, algebra and analytic geometry, whether taught in separate courses or in a unified course, fails to offer the best material to satisfy them. In the first place too much time is spent in these traditional courses on drill in manipulation. As has often been said in recent years, manipulative technique must be regarded as a means to an end, not an end in itself. In so far as it contributes to an understanding of principles it is not only desirable but essential. But beyond that it is of little value to our general student.

In the second place, much of the traditional content is not the most suitable for the satisfaction of our needs. The properties of the conic sections and the methods of analytic geometry in general can not compare in value with the elements of the calculus, most of the work in college algebra is of little use compared with the algebra of finance, etc.

It should be clear, then, that our problem is not primarily one of unification of the traditional material, *it is primarily a problem in the selection of new material.* This material having been selected, some form of unification becomes necessary in order to secure a simple logical structure rather than a mass of isolated topics. The secondary, but equally important problem, relates then to *the organization of the material selected.*

The plea of lack of time will hide my inability to solve either of these problems. Any one who has attempted a solution will realize how difficult they are. But all of us interested in the improvement of mathematical education in this country must contribute what we can toward their solution. As one of my correspondents has pointed out: "On the evolution of a satisfactory textbook depends the continued growth of the movement."

I will not attempt to characterize the ideal textbook of the future. By comparing the earlier ones of 1914 and 1917 with the later ones of 1920 and 1921 some

idea of present tendencies can be gained. It seems safe to say that at present there is a well-defined tendency toward the diminution of most of the traditional material in analytic geometry and toward the inclusion of a considerable amount of calculus. But beyond this it is not safe to indulge in prophecy.

I desire rather, in the remaining time at my disposal, to analyze a bit further some of the difficulties enumerated by our critics—and to offer one or two constructive suggestions.

First of all, I must remove the impression of general dissatisfaction with unified courses, if I have created it by an exclusive insistence on the difficulties. My correspondents include quite as many enthusiastic supporters of the movement as adverse critics. It should be noted that the testimony of those who have used the unified courses is on many points conflicting. While some say that the new type of course is too difficult, others say it is easier than the traditional; while some have found it to provide a poor foundation for subsequent work in mathematics, others have found it to give an excellent foundation; while some claim that it is not well adapted to the varying preparation of students, others find that it is well adapted thereto, etc. And there are many who appear to be firmly convinced that the new type of course is more interesting to the student; that it offers him a far richer and more generally valuable training; that the utility and power of mathematics can be made more apparent to him, and that hence his appreciation of mathematics is increased; that more emphasis is placed on an understanding of principles and less on memorizing formulas, and that it, therefore, is more effective in stimulating thought; and, finally, that the earlier introduction of certain fundamental concepts, such as those of the calculus, extends their period of growth and hence their effectiveness. This last point appears to me of very great importance, since it seems to be in line with a psychological principle which experience has seemed to establish.

Why then this feeling on the part of many, who are theoretically in favor of the new type of course, that the results secured are not as satisfactory as they should be? I wish these gentlemen had given me more detailed specifications as to these unsatisfactory results. The claim that students are less proficient in technique we may pass over, since we ought not to expect them to be as proficient. The claim, however, that the student does not gain as clear-cut concepts under the new as under the older course requires consideration. I wonder if the claim is valid; I wonder if those who make this claim have really used any valid test to find out the facts in the matter. I recently had an illuminating experience that is related to this question. I was teaching a traditional course in analytic geometry and we were on the topic of the point dividing a segment from one point to another in a given ratio. I did my best to explain the meaning of the concept, including the significance of the order of the two given points and of the algebraic sign involved, assigned the lesson and a few days later gave a written quiz on the subject. The members of the class were quite good at using the formula and solving the traditional types of problems. But I passed around slips of paper on which were drawn two parallel straight lines each with a scale

marked on it. There were marked also on each of the lines two points A and B. In addition, on the first of the two lines were marked two points C and D, one of them lying between A and B, the other not between A and B. Under these two lines were written the following questions:

In what ratio does the point C divide the segment from A to B?

In what ratio does the point C divide the segment from B to A?

In what ratio does the point D divide the segment from A to B?

In what ratio does the point D divide the segment from B to A?

Then with regard to the second line, which in addition to the scale divisions had only two points A and B marked, I called for the following:

Mark on the line the point which divides the segment from A to B in the ratio 1 to 3 and label it P. Mark the point which divides the segment from B to A in the ratio 1 to 3 and label it Q. Mark the point which divides the segment from A to B in the ratio -1 to 2 and label it S.

Not more than two or three men in a class of something over twenty could answer these simple questions correctly. These students could do the traditional things well—but they were clearly doing them mechanically. They certainly had no clear idea of the concept involved.

My first constructive suggestion is that we need to develop a new technique of teaching for the new type of course, a technique that is directed toward the new aims. As an essential part of this technique must be developed a new type of test, designed to measure the achievement of students with respect to the fundamental aims of the course. Until such new tests, that are adequate to their purpose, are developed our movement will labor under a severe handicap. For it is manifestly unfair to judge the results secured under the new course by means of tests developed for the traditional course. And is not that what many of us have been doing? Is not here one of the reasons why some of those who have experimented with the new material have found the results discouraging?

I would offer one other suggestion. We need a textbook for the second year's work designed for those who expect to have professional use for mathematics—mathematicians, scientists, engineers and the like; a textbook to follow the first year's work. This second-year course will take up those topics omitted from the first year's work but needed by the prospective engineer let us say—notably analytic geometry, some advanced algebra, and the remainder of the calculus. It will strongly emphasize drill in technique because the student taking this course will need to be technically expert. Such a course would appear to be entirely feasible, if a proper text is provided. To give it from existing texts is awkward and wasteful. Will not some of our authors address themselves to this problem. It seems to me important.

In closing I would merely express my own conviction as to the continued vitality of the movement we have been discussing and my belief that a solution of the various difficulties will be found by a process of successive approximations. Moreover, I believe the time has come when the primary emphasis must be placed on aims, such as those I have attempted to formulate, and on the selection of

material to meet these aims, rather than on unification as such. Unification may not be possible at this stage beyond a certain point; in any case it is a possible means to an end, not an end in itself.

SOME CURIOUS FALLACIES IN THE STUDY OF PROBABILITIES.

By ROBERT E. MORITZ, University of Washington.

Part I.

Quale è'l geometra che tutto s'affige
Per misurar lo cerchio, e non ritruova,
Pensando qual principio ond' egli indige.
Dante, *Paradiso*, canto 33.

It is a strange anomaly that that branch of mathematics, known as the theory of probability, which rests upon the fewest, simplest, and least controvertible fundamental principles, which demands practically no mathematical prerequisites for its pursuit, which throughout occupies itself with innumerable interesting and important problems that even the layman can understand, should be, at the same time, that branch of mathematics which has presented the greatest number of pitfalls to its most illustrious devotees. Indeed it seems that the simplicity, obviousness, and certainty of its basic principles, the easily understood character of its subject matter, and the ease with which it ensnares into error the most skilful dialecticians, are the three outstanding characteristics of this science.

Cardan, as will be shown presently, may be said to have inaugurated the study with a mistaken solution; Pascal, another pioneer of the subject, committed a fallacy in his problem of points involving three players;[1] Leibnitz fell into error in thinking that a throw of twelve with two dice is as probable as a throw of eleven.[2] D'Alembert stumbled time and again when dealing with probabilities. James Bernoulli, in his *Ars Conjectandi*, recorded two erroneous solutions of his nineteenth problem which occurred to him before he obtained its true solution.[3]

Some of the problems that today we expect every schoolboy to solve have been the occasion of serious, and sometimes even acrimonious, contention on the part of mathematicians of the first rank. Take the simple question as to the probability of throwing heads at least once in the course of two tosses of a coin. D'Alembert[4] reasoned that the required probability is 2/3, since there are but three conceivable cases two of which include heads, namely, head on the first toss, or, if head fails to turn up on the first toss then the second toss must show either head or tail, making three possible cases HH, TH, TT. The same sort of reasoning would, of course, make the probability of every future event equal to one-half since there are but two conceivable cases, the occurrence or the non-occurrence of the event.

[1] I. Todhunter, *History of the Theory of Probability*, Cambridge and London, 1865, p. 15.
[2] Leibnitz, *Opera Omnia* (Dutens), vol. 6, pt. 1, p. 217.
[3] I. Todhunter, *l.c.*, p. 69.
[4] I. Todhunter, *l.c.*, p. 258.

D'Alembert [1] considered it obvious that, when on tossing a coin heads have turned up three times in succession, it is more likely that the next toss will result in tail than in head. Bequelin[2] was of the same opinion and concluded hat, when heads have turned up n times in succession, the probability that the next toss will result in head is only $1/(n+1)$ as against a probability of $n/(n+1)$ that the result will be tail.

But little less obvious than the preceding is the following fallacy. If three coins are tossed at once, what is the probability that either three heads or three tails are turned up? Now of three coins at least two must show heads or tails, the probability that the third coin will show like the other two is one-half, hence the probability that all three coins will show alike is 1/2. The fallacy was exposed by Francis Galton.[3]

In the preceding fallacies the errors in logic are easily discerned but there are other problems in which apparently sound logic seems to lead to contradictory results.

Suppose that a poor man and a rich man engage in a game of chance on the following conditions: Each stakes one dollar after which they toss a coin to see who takes the pot. If the rich man wins, the play is ended, for we suppose that the poor man has nothing further to lose, but if the poor man wins they play again the next day, each man doubling his stake. If the rich man wins the second day the play ends, for the poor man will have lost his original stake together with his winnings from the first play and has therefore nothing further to lose. On the other hand, if the poor man wins they continue to play again doubling their stakes, and so on indefinitely.

Common sense tells us that the rich man is sure to win, for no matter how many times in succession the poor man may have won, if he loses the next play he will have lost his original stake together with all of his previous winnings, and, since he has nothing further to lose the play stops automatically, so that he has no chance to recoup. In fact the poor man's losing seems to be a necessary condition of the problem since by hypothesis the play is to continue until the rich man wins.

This problem, known as the martingale, was, according to Cantor,[4] first introduced into mathematical literature by Cardan, who discussed it in his *Practica Arithmeticae* published in 1539. Cardan is said to have shown that the condition of the play imposes a great disadvantage on the rich man. Cantor gives no indication of Cardan's method of proof.

Whitworth seems to hold the opposite view. While he does not discuss the martingale in detail, he announces a principle which clearly covers the case. He says, "If any condition is introduced which requires the play to stop when a certain position is reached, and if that position is more favorable to one player

[1] I. Todhunter, *l.c.*, p. 263.
[2] E. Czuber, Grunert's *Archiv*, vol. 67, p. 8.
[3] *Nature*, vol. 49, p. 365.
[4] M. Cantor, *Geschichte der Mathematik*, vol. 2, Leipzig, 1900, p. 502.

than to the other, the latter is at a disadvantage."[1] The martingale introduces such a condition in that it requires the play to stop when the rich man wins.

Bachelier,[2] one of the most recent writers on probabilities, accepts neither of the foregoing conclusions. Assuming the play to stop when the poor man loses or when he has won n games in succession, he finds the poor man's gain to be $p^n \cdot 2^n - 1$, where p is his probability of winning one game. As n increases indefinitely, this expression approaches ∞, 0, or -1, according as p is greater than, equal to, or less than 1/2. In our case $p = 1/2$, so that the poor man's gain is 0, that is, the poor man has neither an advantage nor a disadvantage over the rich man.

Bachelier's formula is unassailable, yet it leads to a curious paradox if we consider a case in which p is slightly greater than 1/2. Let us assume that the coin used by the players has a tendency when tossed to show heads somewhat more frequently than tails, let us say an average of 51 heads to 49 tails, and that the poor man wins whenever head shows up. Then $p = 51/100$, and Bachelier's formula gives ∞ as the poor man's expectation. This conclusion amounts to saying that no finite sum, however great, would compensate the rich man for the risk he assumes in accepting the conditions of the play provided the coin used in tossing has the slightest tendency to favor the poor man. Common sense assures us that the rich man might well accept the risk on even terms.

A striking illustration of the confusion of thought which has possessed eminent mathematicians when dealing with probabilities is found in Montmort's solution of one of the five problems proposed to him by Nicolas Bernoulli in a letter dated September, 1713, and which Montmort published in the second edition of his *Essai d'analyse sur les jeux de hazard* (Paris, 1714). The third of these problems was as follows:

A deposits a crown. *B* throws a common die and if an even number turns up, he takes a crown, if an odd number turns up he deposits a crown. Then *A* throws the die and if an even number turns up he takes a crown, but deposits nothing if an odd number turns up. Then *B* throws the die again, taking a crown if he throws an even number, depositing a crown if he throws an odd number. Then *A* throws the die again under same conditions as before, and so on indefinitely so long as there remains any sum on deposit. Required the advantage of *A* or *B*.

In his reply to Bernoulli, Montmort states that he had not tried the first two problems, that the last two presented no difficulty, but that it had taken him a long time to solve the third problem, that he had finally come to the conclusion that there would be neither advantage nor disadvantage to *B*, as had also Waldegrave, an English mathematician, who had worked with him on the problem.

The modern reader can only wonder what it was that Montmort and Walde-

[1] W. A. Whitworth, *Choice and Chance*, 5th edition, Cambridge, England, 1901, p. 220.
[2] L. Bachelier, *Calcul des Probabilités*, vol. 1, Paris, 1912, p. 38.

grave worked over, since the problem states explicitly that B has at every trial an even chance of winning or losing.[1]

Bernoulli's fifth problem, which Montmort said presented no difficulty, is the problem which in a slightly modified form has since become celebrated under the name "Petersburg Problem." As first stated the problem admitted of several interpretations, the modified form in which it has become generally known was given it by Daniel Bernoulli, the nephew of Nicolas, who first discussed the problem in the transactions of the Petersburg academy,[2] whence the name of the problem.

In its original form the problem was in substance as follows: B throws a common die. If six turns up on the nth trial, A is to give B a sum of money which is some known function of n. Required B's expectation.

If we denote the sum by $f(n)$, and assume that the play ends when six turns up for the first time, B's expectation is $\Sigma[f(n)\cdot 5^{n-1}/6^n]$. If $f(n)$ is such that $\lim_{n\to\infty}[f(n+1)/f(n)] > 6/5$ or if $f(n+1)/f(n)$ is constant and equal to $6/5$, say $f(n) = (6/5)^n$, B's expectation becomes infinite.

Daniel Bernoulli's modification of the problem makes it read thus: A coin is tossed until head turns up. If this happens on the first trial Peter is to pay Paul 1 crown, if on the second trial 2 crowns, if on the third 4 crowns, and so on, the sum which Peter is to pay Paul when head finally does turn up being double the sum he should have paid had head turned up on the immediately preceding trial. Required Paul's expectation.

This, then, is the Petersburg problem. By Paul's expectation is meant the sum which Paul ought to pay Peter at the outset in order that the play may be fair to both Peter and Paul. The terms of the play are considered fair if, provided the play is repeated indefinitely, neither Peter nor Paul has an advantage.

If an event whose probability is p carries with it a gain P it is easily seen that the expectation of this contingent gain is pP. It follows that the expectation of a series of contingent gains is equal to the sum of the expectations of the separate contingent gains.

In the Petersburg problem the probability that head will turn up for the first time on the nth trial is $1/2^n$, if head does then turn up Paul is to receive 2^{n-1} crowns, hence so far as the nth trial is concerned Paul's expectation is $2^{n-1}/2^n = 1/2$. Now n may have any integral value, hence Paul's total expectation is

$$\sum_{n=1}^{\infty}(2^{n-1}/2^n) = 1/2 + 1/2 + \cdots = \infty.$$

But this conclusion contradicts common sense which assures us that Paul would not pay any considerable sum for the advantage which Peter offers him.

It is of interest to observe that if the foregoing reasoning is valid Paul's expectation will remain infinite even if the conditions of the play are modified in a number of ways, each modification resulting in a distinct disadvantage to Paul.

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[1] I. Todhunter, *l.c.*, p. 134.

[2] "Specimen theoriae novae de mensura sortis," *Comment. Acad. Petrop.*, vol. 5.

1. In the first place the expectation remains infinite whether the initial prospective gain is one crown or any smaller sum, say 10^{-10} crowns.

2. In the second place the expectation remains infinite even though it were stipulated that Paul should receive nothing unless head turned up for the first time at any arbitrarily designated trial, say the one-millionth trial, and if head does not show then he were to receive nothing unless head did not turn up until some other arbitrarily designated trial, say the two-millionth trial, and so on provided only that if head does turn up for the first time on one of the designated trials Paul is to receive the stake attached to that particular trial by the conditions of the problem.

3. The expectation will remain infinite if the stake attached to the nth trial is $2^{n-1}/(kn)$ instead of 2^{n-1} times the initial stake, where k is any constant, however large.

4. Paul's expectation will remain infinite even if the conditions of the play combined all the disadvantages enumerated under 1, 2, and 3.

The proof for this assertion is simple. Paul's expectations of the favorable trials are

$$\text{for the one-millionth trial } \frac{1}{1000000}\cdot\frac{10^{-10}}{k}\cdot\frac{2^{1000000-1}}{2^{1000000}},$$

$$\text{for the two-millionth trial } \frac{1}{2000000}\cdot\frac{10^{-10}}{k}\cdot\frac{2^{2000000-1}}{2^{2000000}},$$

$$\text{for the three-millionth trial } \frac{1}{3000000}\cdot\frac{10^{-10}}{k}\cdot\frac{2^{3000000-1}}{2^{3000000}},$$

$$\cdots\cdots\cdots\cdots,$$

and the sum of these expectations is

$$\frac{1}{1000000}\cdot\frac{10^{-10}}{k}\cdot\frac{1}{2}\left[1+\frac{1}{2}+\frac{1}{3}+\cdots\right]=\infty.$$

In this generalized form of the Petersburg problem Paul would receive nothing unless tail turns up 999,999 times in succession, and his gain, if head turns up on the next trial, is

$$\frac{10^{-16}\cdot 2^{999,999}}{k} \text{ crowns,}$$

a number small at will, say less than 1 crown, since k can be chosen as large as we wish. Yet the theory asserts that Paul's expectation is infinite, that is to say, no matter how large a sum Paul should offer Peter for a single game, Paul would come out winner if the game were repeated on the same terms a sufficient number of times.

Absurd as this conclusion seems, it is an incontrovertible consequence of the fundamental theorem that if A's probabilities of receiving the separate sums $P_1, P_2, P_2, \cdots$, are $p_1, p_2, p_3, \cdots$, respectively, his total expectation is the sum of the separate expectations

$$p_1P_1 + p_2P_2 + p_3P_3 + \cdots.$$

(To be concluded in the next issue.)

THE USE OF AN EXISTENCE THEOREM IN DEVELOPING THE PROPERTIES OF THE SINE AND COSINE.

By H. T. DAVIS, University of Wisconsin.

Introduction. In an interesting section of Professor Osgood's *Funktionentheorie* (Second edition, volume 1, Leipzig and Berlin, 1912, pp. 571–582) many of the properties of the sine and cosine are developed through their definition by means of a differential equation. One method there employed is that of obtaining explicit expansions for the functions by means of substituting series in the equation.

The purpose of the following paper is to show how this same development can be accomplished without the use of series by means of the existence theorem for linear differential equations. The power and elegance of this kind of analysis makes it seem worth while to record a simple example of its use.

The theorem in question is the following:[1]

"A linear homogeneous differential equation, in which the coefficient of the highest derivative is a constant, possesses one and only one integral which, for a preassigned value of the independent variable in whose neighborhood the coefficients of the differential equation remain analytic, assumes a preassigned value and whose first $n - 1$ derivatives assume preassigned values at the same point. The circle of convergence of the ordinary power series development, representing the integral in the neighborhood of such a value, is never smaller than the largest circle within which the power series developments of the coefficients are all convergent."

The power of the method which is here developed attaches to the fact that the theorem just quoted assures us of the uniqueness as well as the existence of the two functions U and V which we shall choose to satisfy the following conditions:

U and V are both solutions of the equation

$$\frac{d^2y}{dx^2} + n^2y = 0, \tag{1}$$

$$\begin{aligned} U(0) &= 1, \qquad V(0) = 0, \\ U'(0) &= 0, \qquad V'(0) = n. \end{aligned} \tag{2}$$

We shall now develop some of the properties of these functions, ultimately to be identified with the sine and cosine, on the basis of this definition.

THEOREM I. $U^2 + V^2 = 1$.

By hypothesis $U'' + n^2U = 0$, $V'' + n^2V = 0$. Multiply the first by V and the second by U and subtract. Then

$$VU'' - UV'' = (VU' - UV')' = 0.$$

Consequently $VU' - UV' = c$, and by making use of conditions (2) we have $UV' - VU' = n$.

[1] L. Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. 1, Leipzig, 1895, p. 25.

We next show that $V' = nU$. Since nU is a solution of the equation, we shall require V' also to be a solution, a fact verified by differentiating the equation and comparing the result with that of substituting V' for y. That V' and nU are actually the same solution follows at once from the uniqueness part of the existence theorem, because

$$V'(0) = n, \qquad nU(0) = n,$$
$$V''(0) = 0, \qquad nU'(0) = 0.$$

It follows similarly that $U' = -nV$ and these results substituted above establish the theorem.

The Oscillation Properties. We shall next prove the following theorem:

THEOREM II. *When n is a real number the functions U and V vanish an infinite number of times along the real axis and their zeros alternate with one another.*

Consider the interval $0 \leqq x < \infty$. We have, if $Z = U'/U$,

$$Z' = \frac{UU'' - U'^2}{U^2} = \frac{-n^2U^2 - U'^2}{U^2} = -n^2 - Z^2.$$

Hence, by integration,

$$U' = -U\int_0^x (n^2 + Z^2)dx,$$

and from the differential equation $U'' = -n^2U$.

If we now suppose that U is positive for all values of x, we are led to a contradiction because U' and U'' would then both be negative which is evidently a sufficient condition that U should vanish at some finite point of the axis of x. From this contradiction we see that U must vanish at least once in the interval.

Moreover, when U vanishes it follows from Theorem I that $V = 1$, and the argument may be repeated for V. Since this may be continued indefinitely, it is seen that U and V must vanish an infinite number of times along the positive, and by similar reasoning, along the negative axis of x.

The second part of the theorem is apparent from the identity:

$$\left(\frac{U}{V}\right)' = \frac{UV' - VU'}{V^2} = \frac{n}{V^2}.$$

Suppose that a and b are two successive zeros of U and that V vanishes at no point in the interval. Then U/V is a continuous function which vanishes at two points and whose derivative must, therefore, by Rolle's theorem, vanish at some point in the interval. But this is clearly a contradiction so that V must vanish in the interval. Since the same argument holds for V, the theorem is seen to be established.

The Addition Formulas. As an example of the application of this method to proving the addition formulas we give the following:

THEOREM III. $U(x + y) = U(x)U(y) - V(x)V(y)$.

[1] If we set $z = x + y$, then by hypothesis the left-hand member satisfies the equation

$$\frac{d^2U}{dz^2} + U = 0,$$

and the additional conditions $U(O) = 1$, $U'(O) = 0$.

Consequently letting $U(x)U(y) - V(x)V(y) = W(x, y)$, it is sufficient to show that W is a function of z, that it satisfies the equation in z, and that $W(O, O) = 1$, $(dW/dz)_{00} = 0$.

A necessary and sufficient condition that W shall be a function of z is that the Jacobian[2]

$$J = \frac{D(z, W)}{D(x, y)} \equiv 0.$$

But $J = U(x)U'(y) - V(x)V'(y) - U'(x)U(y) + V'(x)V(y) \equiv 0$, since $V' = U$ and $U' = -V$.

Also, by means of the theory of transformations,[3] it follows that

$$\frac{d^2W}{dz^2} = \frac{\partial^2 W}{\partial x^2} = U''(x)U(y) - V''(x)V(y),$$

$$= -U(x)U(y) + V(x)V(y).$$

Finally, since

$$\frac{dW}{dz} = \frac{\partial W}{\partial x} = U'(x)U(y) - V'(x)V(y),$$

we have $W(O, O) = 1$, and $(dW/dz)_{00} = 0$.

By virtue of the uniqueness part of the existence theorem, these three conclusions are sufficient to establish the identity.

De Moivre's Theorem. Since the existence theorem applies to complex as well as real functions, it is possible to give a simple proof of De Moivre's theorem.

THEOREM IV. $[U(x) + iV(x)]^n = U(nx) + iV(nx)$.

It is at once clear that the right-hand member of the equation is a solution of the differential equation (1) since it is a linear combination of two particular solutions, and that for $x = 0$ it takes the value 1 and its derivative with respect to x, the value ni.

Consequently it will be sufficient to show that $[U(x) + iV(x)]^n$ is a solution of the differential equation and that this solution and its first derivative at the point $x = 0$ assume the values 1 and ni, respectively.

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[1] In preceding paragraphs U and V stand for $U(nx)$ and $V(nx)$, U' for $dU(nx)/dx$, etc. In the proofs of Theorems III and IV U and V when used alone stand for $U(x)$ and $V(x)$. n may be considered as taken equal to 1, but it should be noted that the n which is introduced in Theorem IV is not the n which is taken equal to 1. Thus we have there $V' = U$ and $U' = -V$, and at the end we have $V'(0) = 1$, although we also have an n which is not supposed to be equal to 1—EDITORS.

[2] Goursat-Hedrick, *Mathematical Analysis*, vol. 1, Boston, 1904, p. 52.

[3] Goursat-Hedrick, *loc. cit.*, p. 65.

If $Y = [U(x) + iV(x)]^n$, then it follows readily that

$$Y'' + n^2Y = (U + iV)^{n-2}[n(n-1)(U' + iV')^2 + n(U + iV)(U'' + iV'') + n^2(U + iV)^2],$$

$$= (U + iV)^{n-2}\{n^2[V^2 - U^2 + U^2 - V^2 + 2i(UV - VU)] + n[V^2 - U^2 - V^2 + U^2 + 2i(UV - VU)]\}$$

$$= 0.$$

Also

$$Y(0) = U(0) + iV(0) = 1,$$

and

$$Y'(0) = n[U(0) + iV(0)]^{n-1}[U'(0) + iV'(0)] = ni,$$

which establishes the theorem.

The Geometrical Interpretation. In order to complete the discussion by showing that the functions U and V are the ordinary cosine and sine, we may interpret them geometrically. We have already shown that $U^2 + V^2 = 1$. Hence if U and V are taken as rectangular coördinates depending upon a parameter, their locus will be a circle of unit radius with center at the origin. It merely remains to give a geometrical interpretation of the parameter nx.

From the relations $V' = nU$ and $U' = -nV$ it follows that $U'^2 + V'^2 = n^2$. Hence $ds/dx = n$ and upon integration, $s + \text{const.} = nx$. This means that the parameter nx is a length of arc measured along the unit circle and, therefore, is also a measure of the angle at the center.

It is interesting to observe that we might have argued as follows: By hypothesis $U'' + n^2U = 0$, $V'' + n^2V = 0$. If we multiply the first equation by V' and the second by U' and subtract, recalling the fact that $UV' - VU' = n$, we have

$$V'U'' - U'V'' = n^2(VU' - UV') = -n^3,$$

or since $U'^2 + V'^2 = n^2$,

$$\frac{(U'^2 + V'^2)^{\frac{3}{2}}}{U'V'' - V'U''} = 1.$$

Hence the radius of curvature of the curve whose parametric representation is $U = U(nx)$, $V = V(nx)$, is 1, which means that the curve itself is a unit circle.

THE EFFECT OF CHANGE OF SCALE ON CURVATURE.

By JAMES K. WHITTEMORE, Yale University.

An important problem of applied mathematics is the determination of an empirical formula to represent the relation between two measured quantities. This problem is fully discussed in Joseph Lipka's *Graphical and Mechanical Computation*,[1] chapter VI. The simplest and most important of the empirical laws are the two-constant laws such as (1) $y = ax + b$, the straight line law;

[1] New York, 1918.

(2) $y = ax^b$, the power law; (3) $y = ae^{bx}$, the exponential law. In each of these equations x and y are corresponding values of the two measured quantities, and a and b are constants. If the form of the law is known, the problem is the determination of the constants. If the form of the law is not known, the first and more difficult problem is the determination of the law best adapted to represent the relation between the measured x, y. The first step in the determination of the form of the law is to plot on ruled paper the points whose coördinates are corresponding numbers x, y. If the plotted points appear to lie nearly in a straight line the law (1) may be tried. It sometimes happens that, even though the plotted points lie approximately in a straight line, the law (1) gives unsatisfactory results. It is then necessary to try a different law. In the choice of a two-constant law other than (1) a most important question is whether the curve joining the plotted points is concave up or down. If the points are approximately in a straight line it is difficult if not impossible to answer this question. "Change of scales from those first used is sometimes advisable in order to bring out the curvature, if any." [1] It is the purpose of this paper to show how different changes of scale affect the numerical value of the curvature, hereafter called simply the curvature, of the graph of a set of points of given coördinates. It is hoped that the results obtained, which are certainly not obvious, may have a practical value in the determination of empirical laws, as well as some general interest.

Consider first a simple example. If we plot the points x, y, which satisfy the equation, $x^2 + y^2 = 1$, with equal scales on the two axes the graph is the unit circle. Suppose the points with the same coördinates to be again plotted with the x scale doubled, the y scale remaining unchanged. Evidently we shall obtain the same curve as in the second plot if we plot with the original equal scales the points of coördinates, $x_1 = 2x$, $y_1 = y$, that is the locus of the equation, $x_1^2/4 + y_1^2 = 1$. The curvature at a point x_1, y_1 of this ellipse is

$$K_1 = \frac{16}{(x_1^2 + 16y_1^2)^{3/2}}.$$

For $x_1 = 2$, $y_1 = 0$, $K_1 = 2$; for $x_1 = 0$, $y_1 = 1$, $K_1 = 1/4$. Clearly doubling the x scale has had quite different effects on the curvature of the circle, unity for every point, at the two points, 1, 0 and 0, 1. Let us next suppose plotted the coördinates of the points of the same circle with the x scale increased in the ratio $m : 1$, m being a positive number greater than one, the y scale unchanged as before. The graph is the same as that obtained by plotting with the original equal scales the locus of the equation, $x_1^2/m^2 + y_1^2 = 1$. If y is not zero,

$$K_1^2 = \frac{m^2}{y^6(m^2 + y'^2)^3},$$

where y and $y' = dy/dx$ refer to the circle, $x^2 + y^2 = 1$, and do not change

[1] *Sheffield Scientific School Sophomore Mathematics*, pamphlet by P. F. Smith, 1921, p. 15.

with m. To find for what points of the circle the curvature increases as m increases from unity we compute $d(K_1^2)/d(m^2)$ for $m = 1$, obtaining

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{y'^2 - 2}{1 + y'^2}, \qquad m = 1.$$

It follows that as the x scale is infinitesimally increased the curvature at a point of the circle increases or decreases as y'^2 is greater or less than 2, that is as the acute angle between the tangent to the circle and the x axis is greater or less than 54° 44′. We proceed to show that the result is the same for any curve.

For any curve the square of the curvature is

$$K^2 = \frac{y''^2}{(1 + y'^2)^3},$$

where $y' = dy/dx$ and $y'' = d^2y/dx^2$. If the coördinates x, y of this curve are plotted with the x scale increased in the ratio $m : 1$ and the y scale unchanged we shall have the same locus as that obtained by plotting with the original equal scales the points of coördinates, $x_1 = mx$, $y_1 = y$. The equation of the new curve is obtained from that of the given curve by replacing x and y in the latter by x_1/m and y_1 respectively. The curvature at x_1, y_1, the point corresponding to x, y, is given by

$$K_1^2 = \frac{y_1''^2}{(1 + y_1'^2)^3} = \frac{m^2 y''^2}{(m^2 + y'^2)^3},$$

since $y_1' = dy_1/dx_1 = y'/m$ and $y_1'' = d^2y_1/dx_1^2 = y''/m^2$. Taking the logarithmic derivative of K_1^2 with respect to m^2, then writing $m = 1$, we find as in the case of the circle

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{y'^2 - 2}{1 + y'^2}, \qquad m = 1.$$

The result is the same for all curves: as the x scale is increased, the y scale remaining unchanged, the acute angle between the tangent to the curve and the x axis decreases, since $y_1'^2 = y'^2/m^2$, and the curvature increases or decreases as y'^2 is greater or less than 2. It is evident that if we consider a definite point of a given curve for which $y' > \sqrt{2}$ and gradually increase the x scale the curvature at the corresponding point will gradually increase until $y_1' = \sqrt{2}$; further increase of the x scale will decrease the curvature. For maximum curvature at a point corresponding to a given point we must choose $m^2 = y'^2/2$, and find that the maximum curvature is given by

$$K_1^2 = \frac{4}{27}\frac{(1 + y'^2)^3}{y'^4} K^2.$$

It is easy to show directly that this maximum K_1^2 is greater than K^2 when y' is greater than 2 and that the ratio of maximum increase K_1^2/K^2 increases with y'^2.

If the equation of a curve is not given but the coördinates of a number of points are known, as in the case where an empirical law connecting corresponding pairs of given numbers is to be found, the effect of change of scale on curvature is the same as if an equation were given. If a number of pairs of coördinates are given and plotted, if it is desired to increase the curvature of the graph at a point where the slope y' is greater than $\sqrt{2}$, and if for other reasons an increase of the x scale is not undesirable, then for maximum curvature the x scale should be increased approximately in the ratio $y' : \sqrt{2}$. If, for example, $y' = 3$, since $3/\sqrt{2} = 2.1$ the x scale should be doubled and $K_1 = 1.4\,K$ approximately.

A discussion similar to the preceding shows that if the y scale is increased in the ratio $n : 1$, the x scale remaining unchanged, the slope y' increases numerically and the curvature increases or decreases as y'^2 is less than or greater than $1/2$. Then if y'^2 is between $1/2$ and 2 an increase of either scale will decrease the curvature while a decrease of either scale will increase the curvature.

We consider finally the effect on curvature of simultaneous changes in both scales. Suppose a curve plotted originally from a given equation or from pairs of given coördinates x, y, with equal scales on the two axes; suppose then the same coördinates plotted with the x scale increased in the ratio $m : 1$ and the y scale increased in the ratio $n : 1$. The second plot may be obtained by plotting with the original equal scales the points whose coördinates are $x_1 = mx$, $y_1 = ny$. For the new curve

$$K_1^2 = \frac{y_1''^2}{(1 + y_1'^2)^3} = \frac{m^2n^2y''^2}{(m^2 + n^2y'^2)^3},$$

since $y_1' = ny'/m$ and $y_1'' = ny''/m^2$. We have

$$\frac{d(K_1^2)}{K_1^2} = \frac{(n^2y'^2 - 2m^2)d(m^2) + (m^2 - 2n^2y'^2)d(n^2)}{m^2n^2(m^2 + n^2y'^2)}.$$

In this equation we set $m = n = 1$ to obtain the change in curvature for infinitesimal changes in scale:

$$\frac{d(K_1^2)}{K_1^2} = \frac{(y'^2 - 2)d(m^2) + (1 - 2y'^2)d(n^2)}{1 + y'^2}, \qquad m = n = 1,$$

an equation which contains the results given before for changes in the x scale alone and in the y scale alone. It is clear that the exact effect on curvature of simultaneous change of both scales is not determined till the relative rate of change of scales is given. We consider two cases: (1) The two scales shall be changed in the same ratio, $m = n$. Since $dm = dn$ we have, for $m = 1$,

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{1}{K_1}\frac{dK_1}{dm} = -1.$$

As the scales increase the curvature decreases as is indeed evident; the rate of change of curvature with respect to scale is exactly equal to the curvature.

(2) The two scales shall be changed in inverse ratio, $m = 1/n$. Suppose for example the x scale to increase, $m > 1$, while the y scale decreases, $n < 1$. In this case the slope, $y_1' = y'/m^2$, decreases numerically; for infinitesimal changes of scale, $m = n = 1$, we have $d(m^2) = -d(n^2)$ and

$$\frac{1}{K_1^2}\frac{d(K_1^2)}{d(m^2)} = \frac{3(y'^2 - 1)}{y'^2 + 1},$$

so that an increasing x scale and a y scale decreasing in inverse ratio will increase curvature as long as $y'^2 > 1$. Suppose that the curvature at a definite point of a given curve for which $y' > 1$ is to be increased by simultaneous inverse changes of scale. For the maximum increase $m^2 = y'$ and

$$K_1^2 = \frac{1}{8}\frac{(1 + y'^2)^3}{y'^3}K^2.$$

We may see directly that this maximum K_1^2 is greater than K^2 when y' is greater than 1, that the ratio of maximum increase K_1^2/K^2 increases with y', and that for values of y' greater than $\sqrt{2}$, as might have been anticipated, this maximum curvature obtained by simultaneous inverse change of scale is greater than the maximum previously found by change of the x scale alone. If again we suppose, for example, $y' = 3$ we obtain the maximum increase of curvature by choosing $m = 1/n = \sqrt{3} = 1.7$ and $K_1 = 2.2\ K$ approximately.

In conclusion we remark that in making a plot of given coördinates for the determination of an empirical law other things besides curvature must be considered. We quote from Lipka:[1] "The scales with which these values are plotted are generally chosen so that the length of the axis represents the total range of the corresponding variable, and so that the line or curve is about equally inclined to the two axes. There is no advantage in choosing the scale units on the two axes equal. Care should be taken not to choose the units either too small or too large; for in the former case the precision of the data will not be utilized, and in the latter case the deviations from a representative line or curve are likely to be magnified. The drawing of a good plot is evidently a matter of judgment." From the results obtained in this paper we know that "if the line or curve is about equally inclined to both axes," and if the plot is not actually a straight line, the curvature will be increased by a decrease in either or both scales, but what change of scale is most advantageous will in the end depend on the precision of the data, the form of the plot, and the size of the paper as well as on a desired magnification of the curvature.

[1] *Loc. cit.*, pp. 122–123.

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

Among the unphilosophical there is generally no very clear notion of what is involved in a mathematical theory; and many of the attacks on this science are attacks, not on mathematics itself, but on the hypotheses on which it happens to operate. Hence arises the distinction, unmeaning in any exact sense, between what is mathematically true and what is true in practice. It is therefore not unlikely that the rather fanciful doctrine of Golden Section, as the most perfect proportion, is by many persons regarded as something proved by mathematics (namely, by the study of the equation $x^2 + x - 1 = 0$), and their unwillingness to apply the doctrine in a wholesale fashion would no doubt be excused on the ground that it is one of those things that are mathematically true only. Readers of the MONTHLY, though innocent of these errors, will be interested in Professor Bennett's article, which goes further and exposes on general ground the impossible character of the theory in the sweeping form in which it is often stated. For notes on the history and literature of Golden Section, compare this MONTHLY, *1918*, 232–235.

The two formulas given by Dr. Weisner, which have a very simple application to a number of elementary summation problems, may well be considered in relation to the operators D and Δ, where $Df(n) = (d/dn)f(n)$ and $\Delta f(n) = f(n+1) - f(n)$. We know that the inverse operators D^{-1} and Δ^{-1} give rise to arbitrary complementary terms, the constants of indefinite integration and indefinite summation. It is also well-known that the operators D, Δ, D^{-1}, Δ^{-1} are commutative, provided the complementary terms are not forgotten. When we pass from the indefinite integral (supposed to exist) to the definite integral with upper limit n, and in like manner to the definite sum, the complementary terms will become fixed, and the results of $D^{-1}\Delta^{-1}$ and $\Delta^{-1}D^{-1}$ will now be found to differ by a linear function of n. The determination of this linear function depends on the lower limits of summation and integration. Dr. Weisner's integration formula (2) may be thought of as giving the results for the lower limits 1 and c; for the left member is of the form $\int_c^n \varphi(n)dn + An + B$. The differentiation formula (1) has the same relation to the change of order of D and Δ^{-1}.

I. THE "MOST PLEASING RECTANGLE."

By ALBERT A. BENNETT, University of Texas.

As has been frequently emphasized, the Golden Section is even richer in interesting associations than was imagined by those who first bestowed upon it this grandiloquent title. We have indeed the regular pentagon, the regular decagon, the regular five-pointed star or pentagram, and related figures depending for their usual construction directly or indirectly upon the Golden Section. Yet

the relation of the segments thus obtained figures numerically in other familiar connections. The proportion, $1 : x = x : (1 - x)$ gives $x^2 + x - 1 = 0$, whence $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{5}$, which enters into the study of the so-called Fibonacci sequence: 0, 1, 1, 2, 3, 5, 8, 13, 21, $\cdots$, where each term after the second is the sum of the two preceding terms. It might seem at first strange that the sum of n terms of a series which is so essentially integral should be represented by an expression involving $\sqrt{5}$, but the following formula is readily verified inductively and constitutes merely the solution of a simple problem in differences:

$$S_n = [(\tfrac{1}{2} + \tfrac{1}{2}\sqrt{5})^{n-1} - (\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n-1}]/\sqrt{5}.$$

The form is more suggestive of the Golden Section, if we alternate signs and write 0, $-$ 1, 1, $-$ 2, 3, $-$ 5, 8, $-$ 13, 21, $\cdots$, where now each term after the second is the difference obtained by subtracting the preceding element from the one preceding that. In this form the solution for the sum is

$$[(-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n-1} - (-\tfrac{1}{2} - \tfrac{1}{2}\sqrt{5})^{n-1}/\sqrt{5}.$$

From the properties of the Golden Section a direct proof that $\sqrt{5}$ is irrational follows easily, since the determination of a common measure for 1 and $-\frac{1}{2} + \frac{1}{2}\sqrt{5}$ can be shown to be impossible owing to the recurrent appearance of this ratio at each step in the classical process.

Despite the wealth of geometrical and arithmetical material that is available in connection with the Golden Section, one sometimes sees in mathematical texts a further statement to the effect that the Golden Section provides segments which when used for the sides of a rectangle result in a "rectangle of the most pleasing shape." One can object to this statement in mathematical texts on three grounds: (1) it is not coördinate with the other propositions, not being mathematical in its content; (2) it could not be proved even it if were true; (3) if not meaningless it is false.

The assertion as to the "most pleasing rectangle" is at its face value incapable of mathematical demonstration, and if intended as a scientific fact can rely only upon psychological grounds. This is not a fatal objection to making a reference to the claims of some psychologists but is a reason for criticizing the abrupt statement sometimes made. It is surely obvious that no mere experimentation can serve to select an irrational number exactly. No matter how many subjects be tested or how many reactions recorded, the determination of an irrational number with absolute precision is out of the question. Two methods of testing occur to mind: (1) A *finite* collection of shapes might be compared, (2) the persons whose esthetic judgments are being invoked might be requested to draw a figure or adjust a mechanism and so select the most pleasing rectangle out of a possible *infinite* number. In either case, at best all might agree on a figure differing but imperceptibly from that given by the Golden Section; and at worst, a distribution of choices might occur with a maximum or mode, near this point. But the statement will be meaningless in the abstract unless it is applicable to all of the numerous concrete instances that come to mind. To be specific, one

may ask what is the most pleasing shape of a sheet of note paper, of a newspaper, of a photograph, of a picture frame, of a postage stamp, etc. The whole question reduces to the character of our esthetic preferences in a matter of this sort. Is there a certain abstract pleasing rectangle which we prefer to see embodied in these various examples, or do practical commercial considerations familiarize us with certain shapes which thereby acquire a certain sanction? The question seems to be one which can be answered readily. One has merely to show that most pleasing proportions in one application are not those in another, and that in some of the most familiar cases at least commercial advantages are sufficient to explain the usual and for these purposes most pleasing proportions.

The proportions of the usual postage stamp when measured from perforation center to center, are $\frac{7}{8}$ in. $\times$ 1 in. These dimensions are selected for the convenience in handling sheets of stamps consistent with ease of handling individually. The factors of the problem are the original size of the sheets to be used, the even decimal number of stamps to a row and to a column desired to facilitate sales, the average size of a clerk's thumb. Long familiarity has accustomed us to finding esthetic satisfaction in this size and shape. Any other size attracts attention by its unfamiliarity, and unless it is an even multiple of the present size looks awkward to the average American. But these are not the proportions of the Golden Section. The American dollar bill and paper money in general show a certain size and shape with which we are all familiar. The dimensions are approximately $3\frac{1}{8}$ in. $\times$ $7\frac{3}{8}$ in. We are accustomed to the handling and folding of bills of this size. When in foreign travel we come upon assorted sizes and shapes, there is nearly always a feeling that the American bill is of more pleasing proportions. In this case again the dimensions are not those of the Golden Section but lie to the other extreme. If a picture that is to be framed should itself show certain proportions, and if the frame is to be of the usual sort to the extent of constituting a uniform border to the picture, it is clear that the exterior dimensions of the frame cannot be in the same ratio as those of the picture. The most pleasing dimensions for a picture frame if such there could be ought to be those presumably fitted to the picture of most pleasing dimensions and therefore the most pleasing rectangle for the exterior of a frame would more nearly approximate a square.

Thus far most of these remarks have been directed to the proportions of a sheet of paper. The most familiar example of such a rectangle is undoubtedly that of a page of a book or magazine. And it is here that simple commercial considerations are most obvious. Changes in shape and in size are both disconcerting, but changes in size are commercially necessary while marked variations in ratio can be and therefore usually are avoided. Small pages are obtained by folding larger sheets, whence the terms, folio, quarto, octavo, duodecimo, etc. The question amounts to one as to how a sheet of paper shall be originally planned so that after these successive foldings the pages shall show sensibly similar shapes, the solution being complicated by the demand for even integral multiples of an inch or at least of a quarter inch. Instead of making one dimen-

sion exactly $\sqrt{2}$ times as long as the other, we can make use of the fact that for most purposes not requiring precision, 2/3 and 3/4 are not sensibly distinct, and are such that their product is 1/2. Thus a sheet, 18×24 when folded gives two sheets, 12×18 of sensibly similar shape, and after another folding, we obtain four sheets of 9×12 which are exactly similar to the original sheet.

By referring to architectural details the folly of speaking of a most pleasing rectangle apart from its use becomes even more evident. The most pleasing rectangle for the cross section of the capital to a column is obviously a square, while the most pleasing shape of a rectangular pilaster has proportions between 1 : 8 and 1 : 15 probably. The most pleasing shape of the tread on a stairway differs again from the most pleasing proportions for a brick or for a fence board, or for a door. The abbreviated regimental colors look awkward to most civilians accustomed to the proportions of the national flag.

There is still another element in the problem. As soon as commercial demands become secondary to the vagaries of fashion, the assumption that there is a most pleasing rectangle even for a given specific purpose becomes doubtful. Exclusiveness and novelty often outweigh the advantages of traditional familiarity. Fancy bricks come in a wide variety of sizes and shapes, ladies' stationery shows continual variations. If there be indeed a most pleasing rectangle why is it that visiting cards whose proportions need be dictated by no considerations other than esthetic show such periodic fluctuations?

Mathematics presents interest enough on its own account, it is called upon in ever increasing measure in the elucidation of natural laws and the furthering of arts. Why must mathematicians persist in seeking extraneous mystical significance in numbers?

II. Note on the Summation of Series.

By Louis Weisner, Columbia University.

The object of this note is to present a method for determining the sum of the first n terms of a series whose nth term is $f'(n)$ or $\int_c^n f(n)dn$, provided that the sum of the first n terms of the series whose nth term is $f(n)$ is known.

If $\varphi(n) = \sum_1^n f(n)$, then $\varphi(n) - \varphi(n-1) = f(n)$.

For most series which occur in practice this difference equation is satisfied for all values of n, with the possible exception of isolated values. If this is the case, and $\varphi(n)$ and $f(n)$ are differentiable functions, we have

$$\varphi'(n) - \varphi'(n-1) = f'(n).$$

If $\varphi'(x) \neq \infty$ when $x = 0, 1, 2, \cdots n$, we may let $n = 1, 2, \cdots n$ in this equation. Adding the n equations thus obtained, we find that

$$\varphi'(n) - \varphi'(0) = \sum_1^n f'(n). \tag{1}$$

For example, from $\frac{1}{4}n^2(n+1)^2 = \sum_1^n n^3$ we get $\frac{n}{2}(n+1)(2n+1) = \sum_1^n 3n^2$.

Again, from $\log \Gamma(n+1) = \log 1 + \log 2 + \cdots + \log n$, we obtain

$$\frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(1)}{\Gamma(1)} = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

Integrating our difference equation between limits c and n, where c is an arbitrary constant for which the integrals exist, we find that

$$\int_c^n \varphi(n)dn - \int_c^n \varphi(n-1)dn = \int_c^n f(n)dn,$$

whence

$$\int_c^n \varphi(n)dn - \int_{c-1}^{n-1} \varphi(n)dn = \int_c^n f(n)dn,$$

or

$$\int_{n-1}^n \varphi(n)dn - \int_{c-1}^c \varphi(n)dn = \int_c^n f(n)dn.$$

Letting $n = 1, 2, \cdots n$, assuming the integrals to have finite values, and adding the n equations thus obtained, we find that

$$\int_0^n \varphi(n)dn - n\int_{c-1}^c \varphi(n)dn = \sum_1^n \int_c^n f(n)dn. \qquad (2)$$

For example from $\frac{1}{4}n^2(n+1)^2 = \sum_1^n n^3$ we get $\frac{6n^5 + 15n^4 + 10n^3 - n}{120} = \sum_1^n \frac{n^4}{4}$.

The reader will have little difficulty in finding other applications.

RECENT PUBLICATIONS.

REVIEWS.

Analytische Behandeling van de Rationale Kromme van den Vierden Graad in een Vierdimensionale Ruimte. By J. Fr. de Vries. The Hague, Martinus Nijhoff, 1922. 8vo. xi + 158 pages. Price 4 guilders.

This book gives an analytical discussion of the rational quartic curve in 4-space, looked upon by the author as the analogon of the twisted cubic in 3-space. The classical analytical treatment of the latter curve [1] is taken as a guide in the study of the C_4^4. The following analysis of the contents of the work gives an indication of its scope:

Chapter I: Some parametric representations of the rational curve of fourth order in 4-space. Conical spaces containing the curve; Chapter II: Simply, doubly and triply tangent spaces, osculating spaces, tangent and osculating planes, tangent lines. Polar space of a point with respect to the curve; Chapter III: The space of the osculating planes, the locus of the tangent lines, the space of the bisecants, polar space. The Plücker numbers; Chapter IV: Quadratic

[1] See, for example, O. Staude, *Analytische Geometrie der kubischen Kegelschnitte*, Leipzig, 1913; or P. W. Wood, *The Twisted Cubic with Some Account of the Metrical Properties of the Cubical Hyperbola*, Cambridge, England, 1913.

spaces containing the curve; Chapter V: Quadratic and cubic involutions on the curve; Chapter VI: Quadratic conical spaces through C_4^4; Chapter VII: Generation of the curve by means of projective systems of spaces and rays; Chapter VIII: The curves C_4^4 through six given points; Chapter IX: Properties of the rational C_4^3 obtained from those of the rational C_4^4 by central projection.

ARNOLD DRESDEN.

The Calculus. By E. W. DAVIS and W. C. BRENKE. Revised edition, New York, The Macmillan Company, 1922. 345 + 65 pages. Price $2.75.

In writing an elementary text on the calculus an author may try to build up a logical treatment of some of the elementary facts as free as possible from intuition and sacrifice manipulative skill to logical rigor, or he may trust to intuition in the more delicate matters and try to develop an intelligent manipulative grasp of the subject.

The first edition of this book followed strongly this second alternative and while it had certain undoubted merits was not altogether a satisfactory text.[1]

The revision by Professors Hedrick and Brenke which has just appeared seems, to the writer, to have cured all the faults of the earlier text and to have given us about as good a book for the average calculus class as one could wish. The authors show an accurate and detailed knowledge of the beginner's difficulties and clearly indicate the right methods of meeting them. The book is not too long and the topics omitted show good judgment—asymptotes and multiple points of curves are not treated at all and the theory of infinite series is given little space. For the first hundred pages the only functions treated are algebraic (mostly polynomials). Reduction formulas do not occupy their usual imposing position and mechanical and formal problem solving receives no encouragement. The treatment of the notion of area as intuitional (which it is not) together with the ambiguity of the integral sign (a now unavoidably defective symbolism) is responsible for the obliteration of the distinction between $D_x^{-1}f(x)$ and $\int_a^b f(x)dx$, which is to be found in practically all the elementary texts. It is responsible for the mental fog which the treatment of the inverse operation as a direct limit usually produces.

Since it is unfair to an author to say nothing but nice things about his book, we must follow the usual custom of casting a few stones of an innocuous sort as certain infelicities of expression. For instance, one would like to know why it it *unfortunate*[2] that positive series whose terms approach zero do *not* converge. Also how can *all*[3] absolutely convergent series be tested by the ratio test or, indeed, by any test? The beginner might imagine they could be successfully tested in this way. It is *not* true, as stated in the foot-note on page 198, that the two definitions of integral are identical. The word *practical* will hardly bear the strain put upon it on page 86 when it is asserted that $x^2 + 5$ and $x^2 + C$ are practically the same thing.

[1] See review in this MONTHLY, *1912*, 202.

[2] p. 270. The beginner might not see the humor unless it is pointed out to him.

[3] p. 263.

The book ends with an excellent formulary of algebra, trigonometry, and analytic geometry with tables and graphs. This last feature though necessary is an eloquent testimonial to the state of preparation of many of our neophytes in the calculus. Let us hope that the day is not far distant when no one will be allowed to enter college without suitable visual training (at least six reels of Charlie Chaplin) and a psychological test certifying that he can name all the brands of automobiles made in the city of Detroit.

M. B. PORTER.

Weather Prediction by Numerical Process. By L. F. RICHARDSON. Cambridge, University Press, 1922. 4to. 12 + 236 pages. Price 30 shillings.

This book is intended primarily for scientific meteorologists, and yet inasmuch as it lays down a suggested program involving preferably international coördination, and certainly much governmental expenditure in Great Britain, it may make a secondary appeal to such legislators, if any there be, as can understand the scientific advantages proposed and weigh these against the financial drawbacks.

The book appears to have three principal purposes. One is to establish the claims of a proposed arrangement of observation stations and a suitable division of labor among them, aiming toward a consistent, inclusive and reasonably refined compilation of scientific data on the weather, for Great Britain at least and preferably for the whole earth. The second purpose is to exhibit and justify a set of partial differential equations in vectorial form which are to account for all the principal meteorological variables. This aim is carried out in a scholarly manner involving the careful weighing of much data and with a breadth of vision of which only an experienced scientist could be capable. The third purpose, one might say, is to explain a set of computing forms by which the partial differential equations may be integrated, on the basis of known meteorological constants and of the periodically observed data secured at the regularly distributed observation stations assumed to exist.

It is not entirely fair to the author of this valuable work to examine only its mathematical features and to ignore the real substance of the discussion. And yet the mathematical form contributes an inherent and important part to the value of the entire undertaking. I will quote some extracts on this point from the preface.

"The extensive researches of V. Bjerknes and his school are pervaded by the idea of using the differential equations for all that they are worth. I read his volumes on *Statics and Kinematics* soon after beginning the present study, and they have exercised a considerable influence throughout it; especially, for example, in the adoption of conventional strata, ... But whereas Prof. Bjerknes mostly employs graphs, I have thought it better to proceed by way of numerical tables. The reason for this is that a previous comparison of the two methods, in dealing with differential equations, had convinced me that the arithmetical procedure is the more exact and the more powerful in coping with otherwise awkward equations. Graphical methods are sometimes elegant when the problem involves irregularly curved boundaries. But the atmospheric boundary, at the earth, nearly coincides with one of the coordinate surfaces, so that the graphs would have no advantages over arithmetic in that respect. . . . The question then arises: how

are we to deal with discontinuities by finite differences? For such purposes graphs have a special facility which numerical tables lack. . . . So probably the most thorough treatment will be reached by tabulating quantities numerically, where they vary continuously, and by drawing a line on the table where there is a discontinuity. The line will be a notification to the computer that one may interpolate up to it from either side, but not across it.

"This investigation grew out of a study of finite differences, and first took shape in 1911. . . ."

From the point of view of the mathematician alone (and any other point of view is out of place here), there is little that is new but still there is much of interest. The facile handling of vectorial partial differential equations, the intelligent application of methods of successive differences, and the reduction of much abstract material to tangible form for immediate computation, put the book beyond the grasp of any reader without mathematical training along these lines. But any one who has had the pleasure of seeing general equations yield numerical results after operations involving many steps of numerical computation will appreciate the problem and will concur with the many suggestive observations of the author. The work is straightforward throughout, and the author has made no attempt to apply ingenious transformations or to make use of invariantive elements.

Two questions at once occur from a scientific point of view: (1) Are the equations adequate? (2) Is the solution capable of any important simplification by making use of the intrinsic characteristics of the problem? The author has not proved and apparently has wisely refrained from claiming to have proved that the equations set up are certain to suffice. The best that can be said seems to be that these equations are at least as adequate as any yet proposed. Much comparison of results with observation will be necessary before the reader can be expected to be convinced that they are definitive. This is almost impossible today because the data that will be provided periodically under the proposed plan are always sadly incomplete with present equipment. If one is to judge by the noteworthy achievements of British, French, and American mathematicians in connection with ballistic problems under the pressure of war demands, one must be convinced that granted all else in the book, the practical handling of the numerical equations will perhaps, even probably, be significantly simplified by a more thorough mathematical investigation of the special features of the problem.

Particular commendation is due to the author for his program of observation areas to be laid off in checkerboard pattern, with alternate data of two sorts to be supplied by alternate stations, an arrangement suggested by the use of central differences. The conscious effort to choose appropriate variables for observation, in such a manner that the subsequent computation may be rendered as simple and effective as possible, shows an appreciation of both sides of the practical problem that experience reveals to be exceedingly rare. Bureaucratic organizations regularly show that too often the observer supplies data of a character determined by accidental considerations, thereby hampering the computer needlessly, or else that the computer demands his material in a form unnecessarily burdening the observer.

One may remark two interesting peculiarities in the text. One is the convenient but not generally current notation for divisions and subdivisions of a

chapter, involving an abandonment of the serial numeration of sections or paragraphs, and the adoption of a method of expression which indicates, at a glance, not only the subdivision number, but also the division and chapter. For example, a subdivision will be marked 3/10/2, to designate the second subdivision, of the tenth division, of chapter three. This is good but not nearly as satisfying as the recently introduced decimal fractional notation. The other minor item worth mentioning is an index of terms with equivalents in Ido, one of the competing "international languages."

The author assumes that the theoretical study and practical forecasting of air conditions is worth the cost and labor involved in the scientific location, equipment and conduct of observation stations, working not independently, but as efficient unit sources of periodic information, to be interpreted in accordance with established physical theory at a central office. This book appears to be the first important expression of this rather generally accepted conviction. Much intelligent scientific effort has been spent in the study of the dynamics of the air. Every year the subject of the prediction of atmospheric conditions becomes increasingly important, while the results capable of simple mathematical treatment continue to be disappointingly meager. Books of this sort, ambitious, original and scholarly, serve to inform and train the novice, and to stimulate fellow scientists.

ALBERT A. BENNETT.

The Fourth Dimension and the Bible. By W. A. GRANVILLE. Boston, Richard G. Badger, The Gorham Press, 1922. 8vo. 9 + 119 pages. Price $1.50.

This work is meant seriously and can be taken in no other sense, however whimsically one might be inclined to approach it at first. One's sympathy with the attempt will depend largely upon one's scientific and religious attitude, and the author admits that he is "on a 'no man's land' exposed to fire from the mathematical trenches on the one side and the theological trenches on the other." For this reason a statement of the content and attitude of the book with quotations in the author's words will doubtless constitute the most satisfactory review.

The preface seeks to establish the reasonableness of the whole inquiry while emphasizing its originality as "practically a virgin field for theological research." The interested reader will do well however to glance at C. F. Bragdon's *Fourth Dimensional Vistas*, New York, A. A. Knopf, 1916. The author first explains why "philosophy and the physical sciences when called to the defence of Christianity have so often proven to be broken reeds," concluding that "it is evident that we cannot reasonably expect that much constructive light will be thrown on Christianity now or in the near future by either philosophy or the physical sciences." On the other hand mathematics "is the only exact science that God has revealed to man and the truths which it contains are the only truths that can be absolutely established through pure reason. Because pure mathematics reveals absolute truth it is part of God himself, for God is the essence of all truth." "That mathematics will ultimately prove to be a valuable aid in solving many of

the perplexing problems connected with our Christian religion is the firm belief of many of our sound thinkers" (with citation from C. J. Keyser).

Approximately half of the book is an exposition of the notion of a fourth dimension and of higher spaces. This portion is in no respect original or scientifically ambitious, but it is thoroughly readable and actually elementary. It is as a whole accurate, calm and logical, with an appeal indeed to one's imagination but none to one's preferences or prejudices. In the remainder of the book, the mathematician is silent, and the discussion can be most generously characterized perhaps, with respect to its philosophical features, as mystical. There comes to mind, as possibly an unfair parallel, the story of Cardan who as a devout and conscientious astrologer is said to have cast the horoscope of Christ, and incidentally to have suffered the consequences of having shocked the medieval church. There seems to be a curious progress from spontaneous analogy to surmise and from surmise to certainty. But the following excerpts may be allowed to speak for themselves:

"There are no restrictions of time, or restrictions of the space of our experience, in dreamland." . . . "Perhaps dreamland is located in four-dimensional space." . . . "If our mental vision is four-dimensional it points to the possibility that our mental or spiritual self is four-(or higher-) dimensional." . . . "In this higher space our grammatical tenses would then have no meaning." . . . "The soul of man, being higher-dimensional, can leave the human body, which is three-dimensional, without doing any violence to the body or leaving any trace on it of the point of emergence." . . . "No image of the soul, which is higher-dimensional, can be secured by material (three-dimensional) means; nor can the human body imprison the soul." . . . "The raising of the dead involves the re-entrance of the souls into their former corporeal habitations."

As to the laws of nature, the author indulges in such remarks as the following:

"This suggests that their (referring to certain snails) external form is the expression of an internal difference due to a right or left twist of their atoms by a four-dimensional force." . . . "A congenital blind and deaf individual . . . could not perceive even any *effect* of those subtle marvelous vibrations that produce light and color and music." "Astronomers have observed the sudden appearance in the heavens of stars and comets and also the apparent vanishing of such heavenly bodies, all in a manner suggesting that they were either entering or leaving our universe. For these and other reasons there has been for some time a suspicion in the minds of many scientists that the total mass of our material universe is not constant. . . . It follows that the principle of the conservation of energy also fails. Numerous instances are recorded in the Bible where new matter or new energy was apparently added to our material universe by supernatural means." "This difficulty (of accepting the miracles) vanishes if instead we look upon the miracles of the Bible as the perfectly logical results of the working out of laws connected with higher spaces."

The author has a chapter on "Spaces as Heavens and Hells." One may not agree with the assertions and surmises made, but in any case, explicit statements are not wanting. Lazarus seems not to have been so fortunate. "Where was the immortal part of Lazarus during the four days in which his body was in the grave, what did he do, what were his experiences? A message from Mars would be of trivial interest to dwellers on earth compared with the importance of the message that Lazarus should have brought back from the regions beyond the grave. It would have answered the great burning question of past, present, and future ages, the question beside which all other questions vanish into insignificance. . . . The reason why Lazarus did not answer this question, did not

relate his experiences between his death and his return from the grave, was because he could not. . . . The impressions he received were therefore received in a space totally foreign to us, impressions which could not be described to us because we have no words, no language which will convey such a description to human (three-dimensional) beings. . . . On being again clothed with his material body his spirit at once became subject to the restrictions and limitations imposed on beings living in our space of three dimensions." But the case is even worse than this. After showing how a three-dimensional material universe might be conceivably transferred bodily into our three-dimensional space and therefore be represented by a material creation, "out of nothing" so far as our space was concerned, and by a similar translation be "annihilated" with respect to our sense perceptions, the writer has the boldness to remark with regard to the limitations of the Holy Scriptures, "The Bible does not and could not tell us how something can become nothing any more than it can tell us how nothing can become something because it was written for human (three-dimensional) beings wholly incapable of comprehending such (higher-dimensional) mysteries."

The author turns readily from mathematics to miracles and again from religious verities to real variables, and does so in a way that is likely to carry conviction to many readers. An orthodox religious attitude, a light and pleasing literary style, and the crisp chapter divisions will retain many as readers who may originally intend merely to glance at this book.

Contents—Preface; Chapter I: The concept of space, 11–17; II: Geometric units in each space, 18–23; III: Of the existence of higher spaces, 24–35; IV: About the supposed inhabitants of other spaces than our own. Flatland. Lineland, 36–46; V: Motion of a point perpendicular to each space, 47–53; VI: Rotation of symmetrical configurations, 54–68; VII: Sections of bodies in one space made by spaces of a lower order. Conservation of matter and of energy, 69–81; VIII: The illusory and the real, 82–85; IX: Spaces as Heavens and Hells, 86–90; X: Creation of our material universe. Evolution, 91–98; XI: Raising of the dead. Sacrament of the Lord's Supper, 99–105; XII: Description of Heaven. Miracles, 106–109; Conclusion, 110–119.

ALBERT A. BENNETT.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 44, April, 1922: "A primary classification of projective transformations in function space" by L. L. Dines, 87–101; "A general theory of limits" by E. H. Moore and H. L. Smith, 102–121; "Substitution groups whose cycles of the same order contain a given number of letters" by G. A. Miller, 122–128; "Boundary values and expansion problems" by R. D. Carmichael, 129–152; "On a theorem in general analysis" by E. W. Chittenden, 153–162.

ANNALS OF MATHEMATICS, volume 23, December, 1921: "Transformations of trajectories on a surface" by J. Lipka, 101–111; "On the structure of finite continuous groups with one two-parameter invariant sub-group" by S. D. Zeldin, 112–117; "On the simplification of the structure of finite continuous groups with more than one two-parameter invariant sub-group" by S. D. Zeldin, 118–121; "The automorphic transformation of a bilinear form" by J. H. M. Wedderburn, 122–134; "A direct determination of the minimum area between a curve and its caustic" by O. Dunkel, 135–140; "The Poisson integral and an analytic function on its circle of convergence" by A. Arwin, 141–143; "Systems of circuits on two-dimensional manifolds" by H. R. Brahana, 144–168; "Two generalizations of the Stieltjes integral" by P. J. Daniell, 169–182.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 28, October, 1922: "Cremona transformations and applications to algebra, geometry, and modular functions" by

A. B. Coble, 329–364; Reviews by S. Lefschetz of F. Severi, *Vorlesungen über algebraische Geometrie* (Leipzig, 1921), 365–366; by P. Field of A. E. H. Love, *Theoretical Mechanics* (3d ed., Cambridge, 1921), 366; by F. L. Hitchcock of A. Naess, *Zur Theorie der Triaden* (Kristiania, 1921), 366–367; by J. B Shaw of C. Isenkrahe, *Untersuchungen über das Endliche und das Unendliche* (Bonn, 1920), 367; by R. B. McClenon of T. L. Heath, *The Copernicus of Antiquity (Aristarchus of Samos)* (London and New York, 1920), 368; by J. B. Shaw of P. Appell, *Eléments de la Théorie des Vecteurs et de la Géometrie Analytique* (Paris, 1921), 368; Notes, 369–392; New publications, 373–376.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 57, October, 1922: Review of É. Picard, *Traité d'Analyse*, volume 1, 3d ed. (Paris, 1922), 353–354; "Sur quelques transformations d'équations aux dérivées partielles" (to be continued) by E. Goursat, 370–384—November: Review by H. Andoyer of Bureau des Longitudes, *Annuaire pour l'an 1922*, 385–389; "Sur quelques transformations d'équations aux dérivées partielles" (conclusion) by E. Goursat, 390–403; "Deux leçons sur certaines équations fonctionnelles et la géométrie non-euclidienne" (to be continued) by E. Picard, 404–416.

L'INTERMÉDIARE DES MATHÉMATICIENS, second series, volume 1, May–June, July–August, 1922: Fifty questions are proposed or re-proposed and thirty-two are answered. The death of H. Brocard, "un des collaborateurs les plus fidèles et les plus érudits du journal," is announced (cover).

MATHEMATICAL GAZETTE, volume 11, December, 1922: "Approximate integration" by A. Buxton, 181–187; "The complete angle and geometrical generality" by D. K. Picken, 188–193; "Isochordic circles related to the triangle" by W. W. Taylor, 194–199; "Some incidental writings by De Morgan," 200–203; Mathematical notes, 204–208; Review by E. H. Neville of Weatherburn's *Elementary Vector Analysis*, 209; Review by H. T. H. Piaggio of Griffin's *Introduction to Mathematical Analysis*, of Passano's *Calculus and Graphs* and of Phillips' *Differential Equations*, 210–211. ["In America a wider range is covered and a broad general outline obtained. The price that has to be paid for this is the sacrifice of much of the mental training obtained by working problems."]

MATHEMATICS TEACHER, volume 15, November, 1922: "The case for general mathematics" by W. D. Reeve, 381–391; "Errors in computation and the rounded number" by H. Rice, 392–404; "The constitution of algebraic abilities" by E. L. Thorndike, 405–415; "Romance in science" by Bessie I. Miller, 416–422; "Mathematics of the calculating machine" by L. L. Locke, 423–428; "'Steradians' and spherical excess" by G. W. Evans, 429–434; and ten pages of discussion, news, notes and research—December: "Non-euclidean geometry" by W. H. Bussey, 445–459; "Study of mathematics under individual system" by Mary M. Reese, 460–466; "Problems concerning the teaching of secondary mathematics" by A. Davis, 467–477; "Future development of mathematical education" by C. N. Moore, 478–483; "Function concept in H. S. mathematics" by J. M. Kinney, 484–495; and twelve pages of discussion, news and reviews.

MESSENGER OF MATHEMATICS, volume 52, May, June, 1922. "Factorization of N & $N' = (x^n \mp y^n) \div (x \mp y)$, &c. [when $x - y = n$]" by A. Cunningham, 1–32.

NATURE, volume 110, September 16, 1922: Notices of A. Dakin, *Practical Mathematics* (London, 1921) and of J. Haag, *Cours complet de mathématiques spéciales*, volume 2, *Géométrie* (Paris, 1921), 375; "The theory of numbers" by G. H. Hardy (Presidential address delivered to Section A of the British Association at Hull, September 8, 1922), 381–385—September 23: Notice of F. F. P. Bisacre, *Applied Calculus* (London, 1921), 411; "Summary of the theory of relativity" by H. J. H. Piaggio, 432–434—October 14: "Bergson and Einstein" [review of H. Bergson, *Durée et Simultanéité* (Paris, 1922)] by H. W. Carr, 503–505—October 21: "Mersenne's numbers" by G. H. Hardy, 542 [correction of the statement made in the presidential address (see above), to the effect that for $n = 137$ it was not yet known whether $2^n - 1$ is prime or composite; the note refers to A. Gérardin's article in *Comptes Rendues du Congrès des Sociétés Savantes*, 1920, 53–55, and to this MONTHLY, *1921*, 380. In the address $n = 139$ should accordingly be substituted for $n = 137$, throughout.]—October 28: Notice of volume 20 (1922), *Proceedings of the London Mathematical Society*, 570; "Relativity and physical reality" by A. A. Robb, 572.

PHILOSOPHICAL REVIEW, volume 31, September, 1922: "7 + 5 = 12" by G. W. Cunningham, 495–504—November: "7 + 5 = 12" by B. Bosanquet, 593–598 [reply to the discussion in the preceding number].

PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES OF THE U. S. A., volume 8, October, 1922: "New properties of all real functions" by H. Blumberg, 283–288; "Generalized limits in general analysis" by C. N. Moore, 288–293.

REVISTA MATEMATICA HISPANO-AMERICANA, volume 4, September–October, 1922: "Prime numbers in arithmetic progression" by E. Landau, 113–128; Appreciation of Camille Jordan by C. de la Vallée Poussin, 129–131; Review of C. P. Steinmetz' *L'Industrie Electrique* by J. A. Perez del Pulgar, 134–135, and fourteen pages of short notes, problems and solutions.

REVUE DE MATHÉMATIQUES SPÉCIALES, volume 33, November, 1922: "On the integrals $\int \frac{dx}{(ax^2+bx+c)^n}$, $\int \frac{P(x)dx}{(ax^2+bx+c)^n}$ " (to be continued) by R. Dontot, 313–315; Solutions of problems in algebra, analytic geometry and physics, 316–324, 330–332; Examination questions, 324–329, 333–336; New problems, 336.

SCIENCE, volume 56, November 10, 1922: "The order of scientific merit" by J. McK. Cattell, 541–547—November 17: Review by H. Blumberg of E. W. Hobson, *Theory of Functions of a Real Variable* (2nd ed., vol. 1, Cambridge, 1921), 574–575; Review by H. L. Rietz of O. Veblen, *Analysis Situs* (New York, 1922), 575.

TÔHOKU MATHEMATICAL JOURNAL, volume 21, October, 1922: "A general view of the theory of summability" by S. Takenake, 193–221; "Notes sur la géométrie du tétrahèdre" by V. Thébault, 222–233; "Notes on differential geometry in non-euclidean space" by T. K. Kubota, 234–243; "New demonstration of Euler-Maclaurin sum formula" by C. Jordan, 244–246; "Note on the theory of approximation of irrational numbers by rational numbers" by K. Kurosu, 247–260; "Beweise einiger Sätze über Eiflächen" by T. Kubota, 261–265; "A mathematical game" by S. Fukazawa, (in Japanese) 265–270; "Pentaspherical geometries in non-euclidean space, III" by T. Takasu, 271–309; "Plane algebraic curves invariant under a quadratic Cremona transformation" by A. Emch, 310–326; "Projective generalization of some theorems on algebraic curves and surfaces" by T. Takasu, 327–348; "On the convergencies of series of functions" by T. Matsumoto, 349–355; "Some integral equations in the theory of diffusion of mixed gases" by T. Hayashi, 355–359.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 23, January, 1922 [published November, 1922]: "Relatively uniform convergence and the classification of functions" by E. W. Chittenden, 1–15; "Periodic functions with a multiplication theorem" by J. F. Ritt, 16–25; "Note on Dirichlet and factorial series" by T. Fort, 26–29; "Functions of infinitely many variables in Hilbert space" by W. L. Hart, 30–50; "Prime and composite polynomials" by J. F. Ritt, 51–66; "Some two-dimensional loci connected with cross-ratios" by J. L. Walsh, 67–88; "On transformations with invariant points" by J. W. Alexander, 89–95; "Invariant points in function space" by G. D. Birkhoff and O. D. Kellogg, 96–115.

UNDERGRADUATE MATHEMATICS CLUBS.

All reports of club activities should be sent to E. L. DODD, Williams College, Williamstown, Mass.

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF BROWN UNIVERSITY, PROVIDENCE, R. I.
[*1922*, 77.]

The program for 1922–23 is announced in printed form, as follows:

November 3, 1922: "Huge numbers" by Philip Welch '23; "The Moscow mathematical papyrus and formulas for the volume of a truncated square pyramid" by Amelia Harris '24; "Sophie Kovalevski, a mathematician of Russia" by Dorothy Bundy '24.

December 15: "Mathematics of genetics" by Doris Anthony '24; "Ramanujan, a mathematician of India" by Everard Ketcham '24; "A Diophantine problem" by John Miner, Jr. '25.

January 12, 1923: "Foundations of our geometric notions" by James Pierpont, professor of mathematics, Yale University.

February 16: "Approximate methods for trisecting an angle" by Nellie Stokes '23; "Robert Recorde, a mathematician of England" by Rose Whelan '25; "Mathematical paradoxes" by George Smith '23.

March 23: "Egyptian mathematics" by Arnold Chace, chancellor of Brown University.

April 27: "Leibnitz, a mathematician of Germany" by Sarah Jacobson '23; "Methods of work of mathematicians" by Richard Whipple '25; "The game of Nim" by Evelyn Wiggin, Gr.

May: Picnic.

The officers are: Chairman, Professor Richardson; committee on program, Professor Burgess, Professor Gilman, Mildred Carlen, Sp., Amelia Harris '24, Philip Welch '23, Henry Bodwell '24; committee on arrangements, Mr. MacPherson, Mary Appel '23, Charlotte Perry '25, Clarence Bennett '23, John Miner, Jr. '25.

The Mathematics Club of Cooper Union, New York City.
[*1922*, 24–25.]

For the year 1921–22, the following officers were elected: President, Thomas Peterson '23; secretary, Hector Audino '25; faculty adviser and honorary treasurer, Professor H. W. Reddick. Meetings were held on alternate Tuesdays, with an average attendance of twenty-five. A polyphase duplex slide-rule was awarded by the Club to Fred Buhrendorf '25 at commencement as a prize for the highest average in first-year mathematics.

The following papers were presented:

November 1, 1921: "Einstein's theory of relativity" by Professor Reddick.
November 15: "Magic squares" by Harry Serper '24.
November 29: "The geometry of numbers" by R. D. Irby '24.
December 13: "Survey of ancient mathematics" by Reginald Overton '23.
January 10, 1922: "Perpetual calendars" (with presentation of wooden models to the Club) by Harry Powell '25 and Albert Johns '25.
January 24: "Geometric representation of indeterminate forms" by William Hoffman '24.
February 7: "Non-Euclidean geometries" by David Samson '24.
February 21: "Amusing numbers" by D. A. Lunden, instructor.
March 7: "The trisection of an angle" by Fred Buhrendorf '25.
March 21: "The fundamental theorem of algebra" by George Agins '22.
April 4: "The slide-rule" by Thomas Peterson '23; "The planimeter" by Battista Sola '23.

At the last meeting the following officers were elected for the year 1922–23: President, Peter Kosting '25; vice-president, Barnett Emmerich '23; secretary, Arthur Cook '25.

(Report by Hector Audino.)

Pi Mu Epsilon of Syracuse University, Syracuse, N. Y.
[*1922*, 80.]

The Pi Mu Epsilon Fraternity has now four chapters. These are located at Syracuse University, Ohio State University, University of Pennsylvania, and University of Missouri, the latter having been recently admitted with fifteen charter members.

At the Syracuse Chapter in 1921–22 there were elected to membership one faculty member, seven graduate students, and nineteen undergraduates. In addition to the formal meetings, a picnic and a Christmas party were given. The officers for the year 1921–22 were: Director, Professor W. G. Bullard; vice-director, Professor May Harwood; secretary, Elizabeth Lyons '22; treasurer, Carl Bye '22; librarian, Eunice Davidson '22. The officers for the year 1922–23 are: Director, Professor Mary Harwood; vice-director, Otto Gelormini, Gr.; secretary, Olive Jackway '23; treasurer, Otis Hendershot '23; librarian, Helen Franklin '23.

Papers were read as follows:

November 7, 1921: "History and plans of Pi Mu Epsilon" by Professor E. D. Roe, Jr.
November 28: "The volume of a tetrahedron in terms of its sides" by Helen Houck '22.
April 24, 1922: "Practical problems in maxima and minima" by Eunice Davidson '22; "Factors in the development of interest in America" by Elizabeth Lyons '22; "Calculus and chemical kinetics" by Otto Gelormini, Gr.
May 18: "The monomolecular reaction constant" by Howard Post, Gr.

(Report by Olive Jackway.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known textbooks, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2999. Proposed by M. B. PORTER, University of Texas.

Given n positive numbers: $a_1, a_2, a_3, \cdots, a_n$, then

$$\Sigma a_i \Sigma a_j^{-1} > n^2 \qquad (i, j = 1, 2, 3 \cdots n)$$

unless $a_i = a_j$, for which case equality occurs; show by passing to limits that, if $\varphi(x) > 0$ and is continuous, $\int_a^b \varphi(x)dx \int_a^b \frac{dx}{\varphi(x)}$ takes on its minimum for $\varphi(x) =$ constant over the interval a to b.

3000. Proposed by J. ROSENBAUM, Milford, Conn.

With use of compasses alone, to construct a circle whose area shall be n times the area of a given circle, where n is any positive integer.

3001. Proposed by NATHAN ALTSHILLER-COURT, University of Oklahoma.

In the plane of a given circle a second circle with a given radius is drawn so that the radical axis of the two circles passes through a given fixed point. Find the locus of the center of the second circle.

3002. Proposed by C. N. MILLS, Heidelberg College, Tiffin, O.

The diagonals of any quadrilateral are in length a and b respectively and are inclined at an angle A. Show that the greatest rectangle which can be drawn with its four sides passing through the four corners of the quadrilateral is $\frac{1}{2}ab(1 + \sin A)$.

3003. Proposed by R. M. MATHEWS, Wesleyan University.

The angle PAM rotates around A and meets a line rotating around B in P and M. When M moves along the perpendicular bisector of AB the locus of P is an equilateral hyperbola of which the mid-point of AB is the center. Generalize.

SOLUTIONS

2908 [1921, 326]. Proposed by L. E. DICKSON, University of Chicago.

If f is a homogeneous polynomial in n variables and H is its Hessian determinant, prove that the Hessian of f^2 is cHf^n, where c is a constant.

SOLUTION BY CONSTANCE R. BALLANTINE, Chicago, Ill.

This is a special case of the general theorem that the Hessian of f^m is $cHf^{n(m-1)}$, which may be proved as follows:

$$H(f^m) = \left| \frac{\partial^2 f^m}{\partial x_i \partial x_j} \right| \qquad (i, j = 1, 2, \cdots, n)$$

$$= m^n \left| f^{m-1} \frac{\partial^2 f}{\partial x_i \partial x_j} + (m-1) f^{m-2} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \right| \qquad (i, j = 1, 2, \cdots, n).$$

This determinant may be broken up into the sum of 2^n determinants, one of which is $H(f)$ with every element multiplied by f^{m-1}, and the others each the same as this determinant with the elements in the jth column replaced by $(m-1)f^{m-2}(\partial f/\partial x_i)(\partial f/\partial x_j)$, for one or more values of j. Of these last, all containing more than one such column will be zero; for suppose the jth and kth columns so replaced, then we may factor out $\partial f/\partial x_j$ and $\partial f/\partial x_k$ and leave these columns the same. Thus,

$$H(f^m) = m^n\left\{Hf^{n(m-1)} + (m-1)f^{n(m-1)-1}\sum_{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}C_{ij}\right\},$$

where C_{ij} is the cofactor of $\partial^2 f/\partial x_i\partial x_j$ in H, and where the indices of summation run from 1 to n.

By Euler's identity, if p is the order of f in the variables,

$$\sum_j x_j\frac{\partial f}{\partial x_j} = pf, \qquad \text{and} \qquad \sum_k x_k\frac{\partial^2 f}{\partial x_k\partial x_i} = (p-1)\frac{\partial f}{\partial x_i},$$

$\partial f/\partial x_i$ being a homogeneous polynomial of order $p-1$ in the variables. Thus

$$\sum_i \frac{\partial f}{\partial x_i}C_{ij} = \frac{1}{p-1}\sum_{ik} x_k\frac{\partial^2 f}{\partial x_i\partial x_k}C_{ij} = \frac{1}{p-1}x_jH,$$

since

$$\sum_i \frac{\partial^2 f}{\partial x_i\partial x_k}C_{ij} = \begin{cases} 0 \text{ when } k \neq j, \\ H \text{ when } k = j. \end{cases}$$

Then

$$\sum_{ij}\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j}C_{ij} = \frac{1}{p-1}H\sum_j x_j\frac{\partial f}{\partial x_j} = \frac{p}{p-1}Hf,$$

and

$$H(f^m) = \frac{m^n(mp-1)}{p-1}Hf^{n(m-1)}.$$

In particular, if $m = 2$,

$$H(f^2) = \frac{2^n(2p-1)}{p-1}Hf^n.$$

Also solved by ELSIE J. MCFARLAND, J. R. MUSSELMAN, F. D. MURNAGHAN, and H. L. OLSON.

2910 [1921, 326]. Proposed by DANIEL KRETH, Wellman, Iowa.

The segments formed on the base of a triangle by the perpendicular from the opposite vertex are m and n. The product of the other two sides is p. Compute the two unknown sides and give a simple construction for the triangle.

PARTIAL SOLUTION BY M. L. YÜ, Nanking, China.

COMPUTATION. In the triangle ABC, let AD be the perpendicular on the side BC, $DB = m$, $DC = n$, the unknown sides $AB = x$, and $AC = y$.

Then we have $x^2 - y^2 = m^2 - n^2$ and $xy = p$.

From these two equations,

$$2y^2 = Y_1 = n^2 - m^2 + \sqrt{(m^2-n^2)^2 + 4p^2},$$

or

$$Y_2 = n^2 - m^2 - \sqrt{(m^2-n^2)^2 + 4p^2}.$$

Y_2 renders y imaginary. $y = \sqrt{Y_1}/2$ gives $x = p\sqrt{2/Y_1}$.

CONSTRUCTION.[1] Draw the straight line RQS, making $RQ = m$ and $QS = n$, and the perpendicular

$$PQ = \sqrt{y^2 - n^2} = \left[\sqrt{(m^2-n^2)^2 + 4p^2} - (m^2+n^2)\right]^{1/2}.$$

Join P and R and P and S and then PRS is the triangle required.

DISCUSSION. In order that there be a real triangle, $PQ^2 = y^2 - n^2$ must be positive, that is, $\sqrt{(m^2-n^2)^2 + 4p^2} - (m^2+n^2) > 0$, which means that $mn < p$.

[1] This is not what is generally meant by a construction but is merely a formula for computation. The geometric construction of PQ from m, n and p^2 is possible but not simple—EDITORS.

Also solved by T. M. BLAKSLEE, J. B. REYNOLDS, A. R. NAUER, and F. L. WILMER.

2912 [1921, 326]. Proposed by T. W. JACKSON, Jamestown College, N. D.

Given c, the chord of a circle, determine r the radius, so that $3c$ is equal to the major arc of the circle.

SOLUTION BY A. H. LAUDER, Laramie, Wyo.

Let 2φ be the angle subtended by the minor arc, and let $x = \sin\varphi = c/2r$. Then $\pi - \varphi = 3x$ and after eliminating φ we have

$$x - \sin 3x = 0, \qquad r = \frac{c}{2x}.$$

The first equation has only one positive root different from zero. Solving for this root and inserting its value in the second equation, it is found that

$$x = .759622, \qquad r = .658222c.$$

Also solved by T. M. BLAKSLEE, A. M. HARDING, J. B. REYNOLDS, and F. L. WILMER.

2913 [1921, 326]. Proposed by PAUL CAPRON, U. S. Naval Academy.

Given

$$S(z) \equiv nz\left[1 + \sum_1^\infty (-1)^p \frac{(n^2-1)(n^2-3^2)\cdots\{n^2-(2p-1)^2\}}{(2p+1)!} z^{2p}\right]$$

$$C(z) \equiv 1 + \sum_1^\infty (-1)^p \frac{n^2(n^2-2^2)(n^2-4^2)\cdots\{n^2-(2p-2)^2\}}{(2p)!} z^{2p}$$

show that, for any value of n, $|\cos x| < 1$, $|\sin x| < 1$:

$$\sin(nx) \equiv S(\sin x) \equiv \sin\left(\frac{n\pi}{2}\right) C(\cos x) - \cos\left(\frac{n\pi}{2}\right) S(\cos x);$$

$$\cos(nx) = C(\sin x) \equiv \cos\left(\frac{n\pi}{2}\right) C(\cos x) + \sin\left(\frac{n\pi}{2}\right) S(\cos x).$$

SOLUTION BY THE PROPOSER.

I. Let $z = \sin x$ and $y = \sin nx$, and let $dy/dz = y'$, $d^2y/dz^2 = y''$. Then $y' = n\cos nx/\cos x$, $y'' = (-n^2\cos x \sin nx + n\cos nx \sin x)/\cos^3 x$, so that $(1-z^2)y'' - zy' + n^2y = 0$.

It is readily seen that this differential equation also arises if we let

II. $z = \sin x$, $y = \cos nx$,

III. $z = \cos x$, $y = \sin nx$,

IV. $z = \cos x$, $y = \cos nx$.

Now let $y = a_0 + a_1z + a_2z^2 + \cdots + a_kz^k + \cdots$, so that $y' = a_1 + 2a_2z + 3a_3z^2 + \cdots + ka_kz^{k-1} + \cdots$, $y'' = 2a_2 + 3.2a_3z + 4.3a_4z^2 + \cdots + k(k-1)a_kz^{k-2} + \cdots$.

If we substitute these values for y, y', and y'' in the differential equation, we shall have as the coefficient of z^k, $(k+2)(k+1)a_{k+2} - (k^2-n^2)a_k$, and this coefficient must vanish identically. Hence

$$y = a_0C(z) + \frac{a_1}{n}S(z).$$

This is the law of the series that satisfies the differential equation $(1-z^2)y'' - zy' + n^2y = 0$.

Sets of corresponding values sufficient to complete the definition of the series for each of the particular cases I, II, III, IV are exhibited in the following table; from them and the law above, the required developments are found:

x	Case	z	$a_0(=y)$	$a_1(=y')$
0	I	0	0	n
0	II	0	1	0
$\pi/2$	III	0	$\sin\frac{n\pi}{2}$	$-n\cos\frac{n\pi}{2}$
$\pi/2$	IV	0	$\cos\frac{n\pi}{2}$	$n\sin\frac{n\pi}{2}$

It is interesting to develop other elementary functions in this way; for instance, to develop $\tan nx$ into a series of powers of $\tan x$, from the equation $(1+z^2)y''+2(z-ny)y'=0$.

2923 [1921, 392]. Proposed by C. N. SCHMALL, New York City.

The corner of a page of a book is turned down in such a manner that the triangle formed has a constant area. Show that the locus of the corner is an oval of the curve,

$$r^2 = a^2 \sin 2\theta.$$

SOLUTION BY W. G. HUBERT, College of the City of New York.

Take the corner in its initial position as origin, in its turned down position as $P(r, \theta)$, and let the ends of the crease be Q and R. By identity, the triangles OQR and PQR are equal, and QR is the perpendicular bisector of OP at E, whose coördinates are $r/2$, θ. The constant area K is then equal to the combined areas of triangles OEQ and OER, or

$$K = \tfrac{1}{2}OE(QE+ER)$$
$$= \frac{1}{2}\frac{r}{2}\left(\frac{r}{2}\tan\theta + \frac{r}{2}\cot\theta\right);$$

whence $r^2 = 4K\sin 2\theta$. The constant a is then equal to $2\sqrt{K}$.

Also solved by P. E. BASYE, EDNA E. KRAMER, R. M. MATHEWS, L. C. MATHEWSON, ADELE A. M. MATZKE, A. R. NAUER, E. J. OGLESBY, H. L. OLSON, ARTHUR PELLETIER, H. P. ROBERTSON, J. B. REYNOLDS, CONSTANCE RUMMONS, ANNA SCHIFF, the PROPOSER, and by MOE BUCHMAN, A. L. DUNA, HERMAN HENKIN, T. F. PETERSON, and BATTISTA SOLA of Cooper Union, New York.

2925 [1921, 393]. Proposed by F. V. MORLEY, New College, Oxford, Eng.

A regular polygon of $2n+1$ sides will have only $n-1$ diagonals of different lengths (*e.g.*, the regular heptagon has two distinct diagonals). Call the side of such a polygon a_1, and the $n-1$ diagonals in order of size $a_2 \cdots a_n$. Then if the circumscribed circle has radius unity, $\sum_{i=1}^{n} a_i^2 = 2n+1$; in words, the sum of the squares of the distinct lengths obtained by joining an odd number of regularly spaced points on a unit circle is equal to the number of such points.

SOLUTION BY A. M. HARDING, University of Arkansas.

Let P be any vertex of the polygon, A_i the ith vertex after P, and $\angle POA_1 = \alpha$. Then $a_i^2 = PA_i^2 = OP^2 + OA_i^2 - 2OP\cdot OA_i\cos i\alpha = 2 - 2\cos i\alpha$. Hence $\sum_1^n a_i^2 = 2n - 2\sum_1^n \cos i\alpha$. Since $\alpha = 2\pi/(2n+1)$ and $2\sin(\alpha/2)\sum_1^n \cos i\alpha = \sin[(2n+1)\alpha/2] - \sin(\alpha/2)$, we have $\sum_1^n a_i^2 = 2n+1$.

Also solved by W. H. ECHOLS, R. M. MATHEWS, L. C. MATHEWSON, H. L. OLSON, J. B. REYNOLDS, F. L. WILMER, M. L. YÜ.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Mr. E. W. SCHREIBER, Proviso Township High School, Maywood, Ill., a member of the Association, is recording secretary of the Chicago Mathematics Club, which meets monthly to consider questions of mathematical or pedagogical interest.

The twenty-second meeting of the Central Association of Science and Mathematics Teachers was held at Hyde Park High School, Chicago, on December 1 and 2, 1922, under the presidency of Mr. ALFRED DAVIS, Soldan High School, St. Louis. Mr. E. L. THOMPSON, Township High School, Joliet, Ill., is vice-chairman of the Mathematics Section. The program of the Mathematics Section was as follows: "Organization of secondary mathematics" by W. W. HART, the discussion being opened by J. M. KINNEY; "Consistency in grading mathematics papers" by E. J. MOULTON; "The slide rule" by W. W. GORSLINE; "Preparation of teachers of mathematics for Junior High School" by J. R. OVERMAN; "Inspection of some old mathematics manuscripts from Armour Institute and other sources" by M. J. NEWELL. All the men mentioned are members of the Mathematical Association of America.

Mr. RALEIGH SCHORLING of the Lincoln School, New York, has been appointed principal and head of the mathematics department of the new model high school at the University of Michigan.

Announcement is made that the annual summer meeting of the Association for 1923 will be held at Vassar College, Poughkeepsie, N. Y., beginning Wednesday afternoon, September 5, and continuing through Wednesday and Thursday. The Thursday afternoon session will be held jointly with the American Mathematical Society whose meetings will continue through Friday.

The following reports of Summer Sessions to be held in 1923 have been received.

University of Chicago, June 18–August 31. In addition to the usual courses in Trigonometry, College algebra, Plane analytic geometry, and Calculus, the following advanced courses are offered. By Professor E. J. MOULTON: Advanced calculus. By Professor J. W. A. YOUNG: Selected topics in mathematics. By Professor A. F. CARPENTER: Solid analytic geometry. By Professor L. J. MORDELL: Theory of definite integrals; Analytic theory of numbers. By Professor A. C. LUNN: Theory of sound; The theory of relativity. By Professor L. E. DICKSON: Arithmetic and algebra of hypercomplex numbers; Differential equations from the standpoint of Lie. By Professor F. R. MOULTON: Analytic mechanics; Theory of functions of the complex variable. By Professor E. J. WILCZYNSKI: Theory of linear differential equations. By Dr. MAYME I. LOGSDON: Higher plane curves.

Columbia University, July 9–August 17. In addition to the usual courses in Logarithms and trigonometry, Solid geometry, College algebra, Analytic geom-

etry, and Calculus, the following courses are offered. By Professor DUNHAM JACKSON: Theory of functions of real variables; Calculus of variations. By Dr. K. W. LAMSON: Differential equations. By Dr. JESSE DOUGLAS: Differential geometry. By Professor W. B. FITE: Theory of infinite series. Each of these courses meets five times per week.

Cornell University, July 7–August 17. By Professor VIRGIL SNYDER: Course for teachers; Projective geometry. By Professor C. F. CRAIG: Analysis. The following reading and research courses are also offered. By Professor SNYDER: Algebraic curves and surfaces. By Professor F. R. SHARPE: Applied mathematics. By Professor W. B. CARVER and Professor F. W. OWENS: Foundations of geometry and problems in synthetic geometry. By Professor D. C. GILLESPIE and Professor W. A. HURWITZ: Advanced analysis. By Professor C. F. CRAIG: Functions of a complex variable. By Professor W. L. G. WILLIAMS: Theory of algebraic forms.

University of Illinois, June 18–August 11. By Professor R. D. CARMICHAEL: Elliptic functions. By Professor A. R. CRATHORNE: Mathematics of statistics. By Professor A. J. KEMPNER: Functions of a complex variable. By Professor G. E. WAHLIN: Theory of equations and determinants. By Dr. H. R. BRAHANA: Projective geometry. By Dr. GEORGE RUTLEDGE: Advanced calculus.

University of Michigan, June 25–August 17. Besides the usual elementary courses, the following more advanced courses are offered. By Professor W. B. FORD: Advanced calculus. By Professor L. C. KARPINSKI: History of mathematics. By Professor T. R. RUNNING: Graphical methods. By Professor PETER FIELD: Vector analysis. By Professor J. W. BRADSHAW: Geometry of pictorial representation. By Professor H. C. CARVER: Theory of probability. By Professor T. H. HILDEBRANDT: Functions of a complex variable. By Professor C. J. COE: Differential equations. By Professor R. B. ROBBINS: Finite differences.

University of Pennsylvania, July 2–August 11. In addition to courses in Solid geometry, Plane trigonometry, College algebra, Analytic geometry, and Calculus, the following courses are offered. By Professor H. H. MITCHELL and Dr. J. D. ESHLEMAN: Advanced plane and spherical trigonometry. By Professor M. J. BABB: Review of college mathematics. By Professor G. G. CHAMBERS: Advanced calculus. By Professor H. H. MITCHELL: Theory of algebraic numbers. By Professor J. R. KLINE: Foundations of geometry.

University of Wisconsin, June 25–August 3. Besides the usual elementary courses, the following more advanced courses are offered. By Mr. O. H. RECHARD: Differential equations. By Professor L. W. DOWLING: Projective geometry, Higher geometry. By Professor W. WEAVER: Vector fields, Restricted theory of relativity. By Professor C. S. SCHLICHTER: Theory of probabilities. By Mr. E. B. MILLER: Theory of equations and determinants. By Professor ARNOLD DRESDEN: College geometry for teachers, Elliptic integrals, Algebra of matrices. By Mr. H. T. DAVIS: Theory of numbers.

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THE DEVELOPMENT OF MATHEMATICS IN BOHEMIA.

By QUIDO VETTER, University of Prague.

Translated from the Italian[1] and edited with notes by

RAYMOND CLARE ARCHIBALD, Brown University.

In the following observations on the development of mathematics in the past, I must consider also astronomy since these two sciences, especially with us, have advanced together. And wishing to show foreigners the scientific movement in a nation little known up to the time of the Great War in the wide circle of an international public, I must sketch at least roughly the surroundings in which it developed. Bohemia, in Central Europe, is surrounded not only by mountains but also by Germans and Magyars, who were quite too frequently enemies of the Czechs. Hence life in this country was fostered as much through the effort of the Czech nation as by the influence of its neighbors.

In the Middle Ages the Czechs knew how to create a strong state which stood at the height of culture of the time. In 1348 the Emperor Karel IV, Czech in origin but educated in Paris (the seat of the famous university), on his return from Italy, where several fine universities flourished, founded the University of Prague which he tried to organize on the models of France and Italy; it was the first university in Central Europe. From that time also our scanty record of the study of the sciences in Bohemia begins. In the Faculty of Arts three professors were named for the Trivium, and four for the Quadrivium that is for the mathematical sciences.[2] Unfortunately their names are not preserved; it is probable that among them were some teachers from Italy. We know, however, the mathematical works which the professors followed in their academic courses. They are the *Algorismus* of Sacrobosco,[3] the six books of Euclid's Elements, the

[1] "Lo sviluppo della matematica nella Boemia" by Q. Vetter, *L'Europa Orientale*, Rome, vol. 2, February 15, 1922, pp. 99–104. This periodical is a monthly issued under the auspices of the Istituto per l'Europa Orientale, founded at Rome (Via Nazionale 89) in 1921, to develop and diffuse, by purely scientific methods, studies relative to Eastern Europe. Among those directing the Institute's work is the scholar Dr. Aurelio Palmieri, a remarkable linguist, ever cordially helpful to inquirers. An excellent library is being built up.

Doctor Vetter has also published "La storia della matematica presso i Cechi" in *Archivio di Storia della Scienza*, vol. 2, June, 1921, pp. 199–201. In the present article he refers to the work of J. Smolík (1863) telling of Bohemian mathematicians before 1750, and to Father Vydras's history (1778) as his chief sources of information. Another useful publication is F. J. Studnička, *Bericht über die mathematischen und naturwissenschaftlichen Publicationen der k. Böhm. Ges. der Wissenschaft während ihres Bestehens*, Prague, 1884. The footnotes I have added contain a number of further references to the literature of the subject; ten of these references were kindly furnished to me by Doctor Vetter. The two footnotes wholly by Doctor Vetter have his name attached to them.

For assistance in the translation I am much indebted to Miss ANNE HATHAWAY of the New York Public Library.

[2] In the Middle Ages the Trivium was the lower division of the liberal arts, comprising grammar, rhetoric, and logic; the higher division was the Quadrivium comprising the mathematical sciences (arithmetic, geometry, astronomy, and music).

[3] Sacrobosco, Latinized form of Holywood; John of Holywood, an Englishman, who flourished in the thirteenth century.